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Polarized Electron Scattering on Spin Zero and Polarized Spin-1/2 Targets: Deep Inelastic Scattering, Elastic Electron–Muon Scattering, and Elastic Electron–Nucleon Scattering

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A covariant formulation is developed and used to derive cross sections for the analysis of experiments in which polarized electrons (muons) are scattered from spin-zero and from polarized spin-1/2 targets. The analysis is based upon the single virtual photon representation of the electromagnetic interaction, initially, neither high-energy nor low-energy approximations are made so that one may derive results in which the orientation of the polarization vectors of the interacting particles changes as a result of the scattering. The general formulation is valid for all polarization configurations for the electron and nucleon in deep inelastic scattering, and for all polarization configurations for the initial and final state particles in elastic scattering. From the general covariant results, specific cross sections are derived for deep inelastic scattering as well as elastic scattering of electrons on muons, nucleons, and spin zero targets. In the latter case, the actual polarization vector for the scattered electron is determined. In the other cases discussed, this vector may be obtained from the cross sections. In addition, a method is presented for defining covariant cross sections, and this method is used to obtain results in the center-of-mass system as well as the laboratory system. Furthermore, explicit cross sections for virtual photon absorption are derived. Finally, in the appendices, an alternative method for the evaluation of traces is given as well as a discussion of the relativistic limit.

1. INTRODUCTION

In the early part of the century Rutherford (1911) presented a theoretical picture of nature that stands as one of the major milestones in the evolution of the physical description of matter. This early picture was confirmed by experiments in which low-energy α and β particles were

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scattered from atoms to provide a startling description of their internal nuclear structure. Throughout subsequent years, these theoretical and experimental ideas were developed so as to provide a clear understanding of the internal structure of the nucleus. Today high-energy electrons are hurled from giant machines like thunderbolts from Zeus to act as probes into the deep internal structure of the nucleons.

At present, techniques have been developed to the point where polarized beams of the electrons can be used to bombard polarized targets (Alguard et al., 1976a, b). Experiments involving the collision of polarized particles are of current interest (Hand, 1977) because they provide valuable information about the structure functions which characterize the fundamental nature of the electromagnetic interaction. The basic interaction which unifies all of these processes over an exceedingly wide range of interaction energies is one mediated by a single spacelike virtual photon. It is my purpose in this paper to present a unified and covariant formulation to aid in the interpretation of experimental polarization phenomena which are associated with this basic interaction. This formulation is valid for all polarization configurations which can occur for the initial and final state particles in the elastic scattering of electrons from spin-zero and spin-1/2 targets; furthermore, it applies as well to the case of deep inelastic scattering where the initial electron and nucleon have arbitrary polarization and where the polarization of the scattered electron is detected. Since the formulation is developed in a covariant manner, it can be used to illustrate the common features of polarization phenomena which occur as a result of the single virtual photon exchange and which are observed from nonrelativistic to ultrarelativistic energies. Of particular interest are those phenomena which are associated with a change in the orientation of the electron's polarization and, in the case of elastic scattering, the target's polarization. Since these effects decrease with increasing interaction energy, their detection at high energy represents a clear challenge to experimental technique and precision.

I begin in Section 2 with a description of the kinematical variables and coordinate systems which are useful for the representation of polarization phenomena. Also in this section, I discuss how one defines various covariant cross sections which provide the direct connection between experimental information and theoretical formulation. Section 3 contains the results for the elastic scattering of a polarized electron from a spin-zero target. Both low- and high-energy limits are obtained and discussed in terms of the actual polarization of the scattered electron. The principal development in this paper is found in Section 4 where the polarization effects associated with deep inelastic electron–nucleon scattering are discussed. Also in this section, one finds an interpretation of this process in terms of the absorption by the nucleon of a spacelike virtual photon. In Section 5 elastic scattering of polarized electrons from polarized muons and from polarized nucleons is described as a special case of the previously developed inelastic process. The modifications required to provide a description of the polarization effects associated with the scattered target are given also in this section. Although much of the tedious algebraic reduction, in particular, the evaluation of traces, has been done with the aid of electronic symbolic computational methods (Hearn, 1973), alternative instructive methods for the evaluation of traces maybe found in Garavaglia (1975) as well as Appendix A. Also in Appendix A, some of the longer results occurring in elastic scattering are found. Finally, an alternative procedure to the one used in the main text for obtaining relativistic limits is described in Appendix B.

Useful introductory discussions of polarization phenomena may be found in Beresteskii et al. (1971) and Schwinger (1970). More detailed information may be found in Dombey (1969), McMaster (1961), and Fradkin and Good (1961). Good insight into the processes involving unpolarized particles may be found in Drell and Walecka (1968). Throughout I have used natural units ($\hbar = c = m_e = 1$, $\alpha = e^2$) and the four-vector, γ matrix, and bispinor conventions of Beresteskii et al. (1971).

2. KINEMATICS AND CROSS SECTIONS

In this section, I describe the kinematical variables and their relationships which are used in the subsequent analysis. As well I outline the general method for defining the various cross sections which are used to characterize the scattering situations which are considered. I begin with a description of the scattering process represented in Figure 1. This process represents an interaction in which particles characterized by four-momenta a and b interact as the result of a single virtual photon exchange to yield a particle characterized by four-momentum c and a collection of N particles characterized by four-momentum $D = \Sigma_{i=1}^{N} p_i$. For this process, energy-momentum conservation is represented as

$$q + b = D, \qquad q = a - c \tag{1}$$

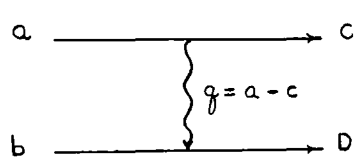

Fig. 1. The single virtual photon exchange in electron–nucleon scattering.

where a typical four-vector is represented as $a = (a^0, \mathbf{a})$ such that $a^2 = (a^0)^2 - \mathbf{a}\cdot\mathbf{a} = m_a{}^2$. Elastic scattering is represented by (1) and the condition $D = d$. The interaction channels are defined according to the invariant quantities

$$4s = (a+b)^2$$

$$4t = (a-c)^2$$

$$4u = (a-d)^2 \tag{2}$$

which satisfy the equation

$$4(s+t+u) = s^2 + b^2 + c^2 + d^2$$

The four-momentum of the virtual photon is denoted by q, which is defined in (1).

In the center-of-mass system, one finds the invariant expressions for energy, momentum, and scattering angle

$$\mathscr{E}_a = \mathscr{E}(s,a,b) = (4s + a^2 - b^2)/4s^{1/2} \tag{3a}$$

$$\mathscr{E}_b = \mathscr{E}(s,b,a), \qquad \mathscr{E}_c = \mathscr{E}(s,c,d), \qquad \mathscr{E}_d = \mathscr{E}(s,d,c)$$

$$|\mathbf{a}| = |\mathbf{b}| = [f(s,a,b)/16s]^{1/2} \tag{3b}$$

$$|\mathbf{c}| = |\mathbf{d}| = [f(s,c,d)/16s]^{1/2}$$

$$\cos\varphi_{ac} = (4t - a^2 - c^2 + 2\mathscr{E}_a\mathscr{E}_b)/2|\mathbf{a}|\,|\mathbf{c}| \tag{3c}$$

with

$$f(s,a,b) = \left[4s - (m_a + m_b)^2\right]\left[4s - (m_a - m_b)^2\right] \tag{3d}$$

or

$$f(s,a,b) = 4\left[(a\cdot b)^2 - a^2b^2\right]$$

In the laboratory system, the corresponding relations are

$$\omega_a = \omega(s,a,b) = (4s - a^2 - b^2)/2m_b$$

$$\omega_b = m_b, \qquad \omega_c = -\,\omega(u,c,b), \qquad \omega_d = -\,\omega(t,d,b) \tag{4a}$$

$$|\mathbf{a}| = f^{1/2}(s,a,b)/2m_b, \qquad |\mathbf{b}| = 0$$

$$|\mathbf{c}| = f^{1/2}(u,c,b)/2m_b, \qquad |\mathbf{d}| = f^{1/2}(t,b,d)/2m_b \tag{4b}$$

In this case, the scattering angle is denoted by θ_{ac}, and it is found from the expression

$$\cos\theta_{ac} = \left[2b^2(4t - a^2 - c^2) - (4s - a^2 - b^2)(4u - b^2 - c^2)\right]/\left[f(s,a,b)f(u,c,b)\right]^{1/2}$$

The principal concern of this investigation is a description of the polarization properties associated with the scattering process depicted in Figure 1. These properties are described with the use of four-vectors which characterize the polarizations of the interacting particles. For a spin-1/2 particle characterized by four-momentum a, the polarization vector $\boldsymbol{\xi}_a$ is defined to be twice the mean value of the particle's spin vector in its rest frame, i.e., $\boldsymbol{\xi}_a = 2\langle\mathbf{S}\rangle$. The components of the vector $\boldsymbol{\xi}_a$ are defined in the rest frame of the particle relative to the unit vector $\mathbf{e}_a$ which is in the direction of the particle's momentum in the frame of observation. The component of $\boldsymbol{\xi}_a$ parallel to $\mathbf{e}_a$ is denoted by λ_a, whereas the component perpendicular to $\mathbf{e}_a$ is denoted by $\boldsymbol{\xi}_{a\perp}$. The polarization vector satisfies the condition $\xi_a{}^2 \leqslant 1$ where equality represents a state of pure polarization and where inequality represents a state of partial polarization. The polarization four-vector associated with $\boldsymbol{\xi}_a$ is a spacelike four-vector obtained from the three-vector as the result of a Lorentz transformation in the direction $\mathbf{e}_a$. This four-vector is represented by

$$s_a = \left[(|\mathbf{a}|/m_a)\lambda_a, \mathbf{s}_a\right] \tag{5}$$

with

$$\mathbf{s}_a = (a^0/m_a)\lambda_a\mathbf{e}_a + \xi_{a\perp}\mathbf{e}_{a\perp}$$

The polarization four-vectors for other spin-1/2 particles are defined in a similar manner.

For the scattering of an electron of unit mass in the initial state $|a, s_a\rangle$ to the final state $|c, s_c\rangle$, the four-vectors s_a and s_c are given in the center-of-mass (c.m.) system as

$$s_a = (|\mathbf{a}|\lambda_a, \mathbf{s}_a)$$
$$s_c = (|\mathbf{a}|\lambda_c, \mathbf{s}_c) \tag{6}$$

with

$$\mathbf{s}_a = \mathscr{E}_a\lambda_a\mathbf{e}_a + \xi_{a\perp}\mathbf{e}_{a\perp}$$
$$\mathbf{s}_c = \mathscr{E}_a\lambda_c\mathbf{e}_c + \xi_{c\perp}\mathbf{e}_{c\perp}$$

The unit vectors $\mathbf{e}_a, \mathbf{e}_{a\perp}$ and $\mathbf{e}_c, \mathbf{e}_{c\perp}$ are defined relative to the coordinate systems S_a and S_c, which are defined as follows: The system S_a is a right-handed orthonormal triad defined by the unit vectors

$$\mathbf{e}_a, \qquad \mathbf{e}_2 = (\mathbf{e}_a \times \mathbf{e}_c)/\sin\varphi_{ac}, \qquad \text{and } \mathbf{e}_1 = \mathbf{e}_2 \times \mathbf{e}_a \tag{7a}$$

The system S_c is defined similarly in terms of the orthonormal triad

$$\mathbf{e}_c, \qquad \mathbf{e}_2, \qquad \text{and } \mathbf{e}_{1'} = \mathbf{e}_2 \times \mathbf{e}_c \tag{7b}$$

Viewed in a direction antiparallel to the momentum $\mathbf{a}$, the vector $\mathbf{e}_{a\perp}$ lies in the plane $(\mathbf{e}_1, \mathbf{e}_2)$ and makes an angle α with the direction $\mathbf{e}_1$. With a similar definition of the angle β between $\mathbf{e}_{c\perp}$ and $\mathbf{e}_{1'}$ in the system S_c, one finds

$$\begin{aligned} \mathbf{e}_c &= \cos\varphi\mathbf{e}_a + \sin\varphi\mathbf{e}_1 \\ \mathbf{e}_{1'} &= -\sin\varphi\mathbf{e}_a + \cos\varphi\mathbf{e}_1 \\ \mathbf{e}_{a\perp} &= \cos\alpha\mathbf{e}_1 + \sin\alpha\mathbf{e}_2 \\ \mathbf{e}_{c\perp} &= \cos\beta\cos\varphi\mathbf{e}_1 + \sin\beta\mathbf{e}_2 - \cos\beta\sin\varphi\mathbf{e}_a \end{aligned} \tag{8}$$

For the elastic scattering in the c.m. system of an electron and a spin-1/2 particle of mass m with the initial state $|b, s_b\rangle$ and with the final state $|d, s_d\rangle$, the polarization four-vectors s_b and s_d become

$$\begin{aligned} s_b &= [(|\mathbf{a}|/m)\lambda_b, \mathbf{s}_b] \\ s_d &= [(|\mathbf{a}|/m)\lambda_d, \mathbf{s}_d] \end{aligned} \tag{9}$$

with

$$\begin{aligned} \mathbf{s}_b &= (\mathscr{E}_b/m)\lambda_b\mathbf{e}_b + \xi_{b\perp}\mathbf{e}_{b\perp} \\ \mathbf{s}_d &= (\mathscr{E}_b/m)\lambda_d\mathbf{e}_d + \xi_{d\perp}\mathbf{e}_{d\perp} \end{aligned}$$

The corresponding unit vectors are given by

$$\begin{aligned} \mathbf{e}_b &= -\mathbf{e}_a, \qquad \mathbf{e}_d = -\mathbf{e}_c \\ \mathbf{e}_{b\perp} &= \cos\gamma\mathbf{e}_1 + \sin\gamma\mathbf{e}_2 \\ \mathbf{e}_{d\perp} &= \cos\delta\cos\varphi\mathbf{e}_1 + \sin\delta\mathbf{e}_2 - \cos\delta\sin\varphi\mathbf{e}_a \end{aligned} \tag{10}$$

where γ and δ are the angles measured from $\mathbf{e}_1$ and $\mathbf{e}_{1'}$, respectively.

The confrontation between theoretical analysis and experimental phenomena is accomplished with the aid of various differential cross sections which may be found from a conventional definition (Källēn, 1964) of the invariant total cross section for the interaction of two particles in the states $|a, s_a\rangle$ and $|b, s_b\rangle$, respectively, and the subsequent production of an n-particle final state where each particle is characterized by a momentum and polarization state $|p_i, s_i\rangle$.

This cross section is defined as

$$\sigma(s,t,u,s_a,s_b,s_c,\dots) = \frac{1}{2f^{1/2}(s,a,b)(2\pi)^{3n-4}} \times \int dp_1 dp_2 \dots dp_n \prod_{i=1}^{n} \delta(p_i^2 - m_i^2)\theta(p_i) \times \delta\left(a+b-\sum_{i=1}^{n} p_i\right)\mathscr{M}(s,t,u,s_a,s_b,s_c,\dots) \tag{11}$$

with

$$\theta(p) = \left[(p^0/\omega)+1\right]/2, \qquad \omega = \left(|\mathbf{p}|^2 + m^2\right)^{1/2},$$

$$\text{and} \quad \mathscr{M}(s,t,u,s_a,s_b,s_c,\dots) = |\langle f|M|a,s_a,b,s_b\rangle|^2 \tag{12}$$

where $\langle f|M|a,s_a,b,s_b\rangle$ is the transition amplitude from the initial to the final state. In the definition (11), one uses the invariant measure in momentum space,

$$\frac{d\mathbf{p}}{(2\pi)^3 2\omega} = \frac{\theta(p)\,dp\,\delta(p^2-m^2)}{(2\pi)^3} \tag{13}$$

One also uses an invariant definition of the flux which is represented as the magnitude of the relative velocity $|\mathbf{v}_a - \mathbf{v}_b|$ in the c.m. system. With the aid of (3), the flux becomes

$$F = f^{1/2}(s,a,b)/2\mathscr{E}_a\mathscr{E}_b \tag{14}$$

Particular differential cross sections may now be obtained from (11). Of special interest is the differential cross section defined formally as

$$\frac{d\sigma}{dt}(s_a,s_b,s_c,s_d) = 4\sigma(s,t,s_a,s_b,s_c,s_d)\delta\left[4t-(a-c)^2\right] \tag{15}$$

For scattering into the solid angle $d\Omega_{ac}$, one finds from (15) for elastic scattering in the c.m. system

$$\frac{d\sigma}{d\Omega_{\text{c.m.}}}(s_a, s_b, s_c, s_d) = -\left(\frac{1}{2\pi}\right)\frac{d\sigma}{dt}(s_a, s_b, s_c, s_d)\frac{dt}{d\cos\varphi_{ac}} \quad (16a)$$

which with (3) becomes

$$\frac{d\sigma}{d\Omega_{\text{c.m.}}}(s_a, s_b, s_c, s_d) = \frac{1}{64\pi s} f^{1/2}(s,a,b) f^{1/2}(s,c,d)\frac{d\sigma}{dt}(s_a, s_b, s_c, s_d) \quad (16b)$$

The corresponding differential cross section in the laboratory system may be found from (4) and (11) in a similar manner, and it becomes

$$\frac{d\sigma}{d\Omega_{\text{lab}}}(s_a, s_b, s_c, s_d) = \frac{-f^{1/2}(s,a,b)f^{3/2}(u,c,b)}{2\pi g(s,t,a,b,c,d)}\frac{d\sigma}{dt}(s_a, s_b, s_c, s_d) \quad (17a)$$

with

$$\begin{aligned} g(s,t,a,b,c,d) = 4\Big\{&(4u-c^2-b^2)\big[2b^2(4t-a^2-c^2)\\ &-(4u-c^2-b^2)(4s-a^2-b^2)\big]\\ &+(4s-a^2+b^2)\big(4u-(m_b+m_c)^2\big)\\ &\big(4u-(m_b-m_c)^2\big)\Big\} \end{aligned} \quad (17b)$$

For the elastic scattering of an electron of unit mass with a particle of mass m, one finds the useful expression

$$\begin{aligned} g(s,t,1,m,1,m) = 128m^2\big[&s^2+st-s(m^2+1)/2\\ &-t(m^2-1)/4+(m^2-1)^2/16\big] \end{aligned} \quad (18)$$

The integration indicated in (11) and (15) when there is a two-particle final state characterized by four-momenta c and d is performed in the c.m. system with the momentum space measure

$$dc\,dd = |\mathbf{c}|^2 d|\mathbf{c}|\,d\Omega_c \frac{dc^2}{2c^0}d\mathbf{d}\frac{dd^2}{2d^0} \quad (19a)$$

and with the aid of the δ functions $\delta(a+b-c-d)=\delta((4s)^{1/2}-c^0-d^0)\delta(\mathbf{c}+\mathbf{d})$. After integration over d^2, c^2, and $\mathbf{d}$, one then integrates over $\cos\varphi_{ac}$ and finally over $|\mathbf{c}|$ with

$$|\mathbf{c}|d|\mathbf{c}|=c^0d^0d(c^0+d^0)/(c^0+d^0) \tag{19b}$$

to find

$$\frac{d\sigma}{dt}(s_a,s_b,s_c,s_d)=\frac{1}{4\pi f(s,a,b)}\mathscr{M}(s,t,s_a,s_b,s_c,s_d) \tag{20}$$

For all of the cross sections so far defined, when s denotes the polarization four-vector for a spin-1/2 particle, averaging over the spin directions in the initial state is accomplished with $s=0$. For a spin-1/2 particle in the final state, the summation over the spin directions of this particle is accomplished with $s=0$ and with the cross section multiplied by 2.

3. ELASTIC ELECTRON SCATTERING ON A SPIN-ZERO TARGET

In this Section, I describe the elastic scattering of an electron on a spin-zero target. This example is simpler than the others that are considered, and it is useful for pointing out the physical effects and the analytical techniques which are similar to those that occur in more complicated interactions.

This analysis is based on the scattering diagram in Figure 1 where an electron initially in the state $|a,s_a\rangle$ interacts with a spin-zero target of four-momentum b and mass m. The scattered electron is represented by the state $|c,s_c\rangle$, and the final state of the target is represented by the four-momentum d. The polarization four-vector s_c represents the polarization which is accepted by the detector. The polarization four-vectors s_a and s_c satisfy the conditions $s_a\cdot a=s_c\cdot c=0$, and $-s_a{}^2\leqslant 1$, $-s_c{}^2\leqslant 1$, where for the second pair equality represents a state of pure polarization. The transition amplitude for this process is represented in momentum space in terms of the electron current $J(a,c)_e$, the hadron current $J(b,d)_h$, and the photon propagator $D_{\mu\nu}(t)$ as

$$\langle c,d|M|a,b\rangle=\alpha J(a,c)_e\cdot D(4t)\cdot J(b,d)_h \tag{21}$$

where

$$J^\mu(a,c)_e=\bar{u}(c)\gamma^\mu u(a)$$
$$J^\nu(b,d)_h=F(4t)p^\nu;\qquad p=2b+q$$
$$D_{\mu\nu}(4t)=\pi g^{\mu\nu}/(t+i\varepsilon) \tag{22}$$

According to the definition (12), one finds

$$\mathcal{M}(s,t,s_a,s_c)=\left[\frac{\alpha\pi F(4t)}{t}\right]^2 p\cdot L(s_a,s_c)\cdot p \tag{23}$$

where the gauge-invariant lepton polarization tensor $L^{\mu\nu}(s_a,s_c)$ is

$$L^{\mu\nu}(s_a,s_c)=\mathrm{Tr}[\rho_c\gamma^\mu\rho_a\gamma^\nu] \tag{24}$$

In this definition, the polarization density matrix for an electron of four-momentum a is defined as

$$(\rho_a)_{ij}=u_i(a)\bar{u}_j(a) \tag{25}$$

such that $\mathrm{Tr}\,\rho_a=2$.

This matrix may be expressed in the covariant form (Michel and Wightman, 1955)

$$\rho_a=\rho_{0a}(1+\not{s}_a\gamma^5) \tag{26a}$$

with

$$\rho_{0a}=(\not{a}+1)/2 \tag{26b}$$

After evaluating the trace in (23), one finds the invariant differential cross section from (20). It is now possible to write this cross section in a form which brings out the physical influence of the interaction on the scattered electron's polarization vector. This is accomplished if one writes (23) in terms of the actual polarization four-vector s_c^f of the scattered electron. The polarization density matrix ρ_c^f associated with the four-vector s_c^f may also be represented in the form (26). When ρ_c is similarly represented, it characterizes the polarization accepted by the detector. Since the probability for detecting the polarization represented by s_c is equal to $\mathrm{Tr}[\rho_c^f\rho_c]$, one can conclude from (24) that ρ_c^f is proportional to $\rho_{0c}\not{p}\rho_a\not{p}\rho_{0c}$ so that $\rho_c^f(1-\not{c})=(1-\not{c})\rho_c^f=0$. With both ρ_c and ρ_c^f represented as in (26) and with the condition $\rho_{0c}{}^2=\rho_{0c}$, it follows that

$$\mathrm{Tr}[\rho_c^f\rho_c]=2(1-s_c^f\cdot s_c)=\frac{2}{A(s,t)_0}\mathrm{Tr}[\rho_c\not{p}\rho_a\not{p}] \tag{27}$$

Using the reduction of $L^{\mu\nu}(s_a,s_c)$ in terms of the gauge-invariant tensors given in (B4), one can show that

$$p\cdot L(s_a,s_c)\cdot p=A(s,t)_0+p\cdot LS(s_a,s_c)\cdot p \tag{28}$$

with

$$A(s,t)_0 = p\cdot L(0,0)\cdot p = 8\left[(s-u)^2 + (m^2-t)t\right] \tag{29}$$

The components of s_c^f can now be found as the coefficients of the components of s_c in the expression

$$s_c^f\cdot s_c = -\,p\cdot LS(s_a,s_c)\cdot p/A(s,t)_0 \tag{30}$$

Returning to (23), one finds that the invariant differential cross section may be written in the form

$$\frac{d\sigma}{dt}(s_a,s_c) = \left[\frac{\alpha\pi F(4t)}{t}\right]^2 \frac{A(s,t)_0}{4\pi f(s,1,m)}\left(1 - s_c^f\cdot s_c\right) \tag{31}$$

Upon evaluation of the trace in (30), one finds the invariant expression

$$s_c^f\cdot s_c = s_a\cdot s_c - \frac{4}{A(s,t)_0}\Big[(4s-m^2-1)(a\cdot s_c b\cdot s_a + b\cdot s_c c\cdot s_a)$$
$$+4tb\cdot s_a(a\cdot s_c + b\cdot s_c) - m^2 a\cdot s_c c\cdot s_a\Big] \tag{32}$$

One may now use (31) and (32) to obtain information about all polarization configurations which can occur for the initial and the scattered electron. If one notes that in the rest frame of the scattered electron $s_c^f\cdot s_c = -\,\boldsymbol{\xi}_c^f\cdot\boldsymbol{\xi}_c$, then he may find the components of the vector $\boldsymbol{\xi}_c^f$ as the coefficients of the components of $\boldsymbol{\xi}_c$. To find these components, one writes s_a, and s_c as in (6) and uses (32) to obtain

$$\lambda_c^f = \left[1 + \frac{16t(4s+m^2-1)^2}{f(s,1,m)A(s,t)_0}\right]\lambda_a$$
$$+\left[\frac{(4s-m^2-1)(4s+m^2-1)}{s^{1/2}A(s,t)_0}\sin\varphi_{ac}\right]\cos\alpha\xi_{a\perp} \tag{33}$$

$$\xi_{c1'}^f = \left(1 + \left\{1 - \frac{[f(s,1,m)+16st](4s-1)}{2sA(s,t)_0}\right\}\frac{32st}{f(s,1,m)}\right)\cos\alpha\xi_{a\perp}$$
$$-\left[\frac{(4s-m^2-1)(4s+m^2-1)\sin\varphi_{ac}}{s^{1/2}A(s,t)_0}\right]\lambda_a$$

$$\xi_{c2}^f = \sin\alpha\xi_{a\perp}$$

with

$$t = \frac{-f(s,1,m)}{16s}\sin^2(\varphi_{ac}/2) \tag{34}$$

where the scattering angle is measured in the c.m. system.

Although (31), (32), and (33) describe all polarization phenomena associated with this simple process, much physical insight may be gained if one considers various limiting results which can be obtained from these equations. Beginning with the relativistic limit in which one neglects the electron's mass relative to the quantity $4s$, one finds from (33)

$$\begin{aligned} \lambda_c^f &= \lambda_a \\ \xi_{c1'}^f &= \cos\alpha\xi_{a\perp} \\ \xi_{c2}^f &= \sin\xi_{a\perp} \end{aligned} \tag{35}$$

This result tells one that in this limit the orientation of the electron's polarization vector relative to its momentum is unchanged as a result of the scattering process.

Additional insight is gained into the effect of the interaction on the electron's polarization vector, if one considers the case when the target's mass is much larger than the electron's mass and energy. In this case the scattering angle φ_{ac} in the c.m. system is the same as the scattering angle θ_{ac} in the laboratory system. With the approximations

$$\begin{aligned} 4s &\approx m(m+2\omega_a) \\ f(s,1,m) &= 4m^2(\omega_a^2-1) \end{aligned} \tag{36}$$

one observes that the components of the polarization vector in (33) become

$$\begin{aligned} \lambda_c^f &= f_1(\omega_a,\theta)\lambda_a + f_2(\omega_a,\theta)\cos\alpha\xi_{a\perp} \\ \xi_{c1'}^f &= f_1(\omega_a,\theta)\cos\alpha\xi_{a\perp} - f_2(\omega_a,\theta)\lambda_a \\ \xi_{c2}^f &= \sin\alpha\xi_{a\perp} \end{aligned} \tag{37}$$

with

$$\begin{aligned} f_1(\omega_a,\theta) &= 1-2/\left[1+\omega_a^2\cot^2(\theta/2)\right] \\ f_2(\omega_a,\theta) &= 2\omega_a\cot(\theta/2)/\left(1+\omega_a^2\cot^2(\theta/2)\right. \end{aligned} \tag{38}$$

For this case, one finds in the relativistic limit $f_1(\omega_a, \theta) \approx 1$ and $f_2(\omega_a, \theta) \approx 0$ so that one again recovers (35). Further insight is obtained from this case if one now considers the low-energy limit in which the energy $\omega_a \approx 1$. In this limit $f_1(1, \theta) = \cos\theta$ and $f_2(1, \theta) = \sin\theta$ so that (37) becomes

$$\begin{aligned}
\lambda_c^f &= \cos\theta\lambda_a + \sin\theta\cos\alpha\xi_{a\perp} \\
\xi_{c1'}^f &= -\sin\theta\lambda_a + \cos\theta\cos\alpha\xi_{a\perp} \\
\xi_{c2}^f &= \sin\alpha\xi_{a\perp}
\end{aligned} \tag{39}$$

In this way, one recovers the well-known result that the quantization direction of the electron's spin does not change as a result of low-energy scattering. At this point it is interesting to note that the result for the special case given in (37) agrees with the result on p. 272 of Beresteskii et al. (1971), where

$$\boldsymbol{\xi}_c^f = \frac{(A^2 - |B|^2)\boldsymbol{\xi}_a + 2|B|^2\mathbf{e}_2 \cdot \boldsymbol{\xi}_a + 2A|B|\mathbf{e}_2 \times \boldsymbol{\xi}_a}{A^2 + |B|^2} \tag{40}$$

with

$$\begin{aligned}
A &= \omega_a + 1 + (\omega_a - 1)\cos\theta \\
B &= -i(\omega_a - 1)\sin\theta
\end{aligned}$$

Further understanding is achieved when one considers the specific differential cross sections which are related to the limiting results already discussed. In the laboratory system, the differential cross section of interest is found from (17) and (31) with $m_a = m_c = 1$, and $m_b = m_d = m$. For this case the relativistic limit is found when one uses the approximations

$$\begin{aligned}
f(s,1,m) &\approx (4s - m^2)^2 \\
s - m^2/4 &\approx (m/2)\omega_a \\
u - m^2/4 &\approx -(m/2)\omega_c \\
t &= -\omega_a\omega_c\sin^2(\theta/2) \\
\omega_c &= \omega_a/\left[1 + 2(\omega_a/m)\sin^2(\theta/2)\right]
\end{aligned} \tag{41}$$

In this limit, the differential cross section in the laboratory system becomes

$$\frac{d\sigma}{d\Omega_{ac}}(s_a, s_c) = \frac{d\sigma}{d\Omega_{ac}}(0,0)\left(1 + \boldsymbol{\xi}_c^f \cdot \boldsymbol{\xi}_c\right) \tag{42}$$

with ξ_c^f given by (35). The cross section resulting from averaging on the spin directions of the initial electron and summing on the spin directions of the scattered electron is

$$2\frac{d\sigma}{d\Omega_{ac}}(0,0)=\left[\frac{\alpha\cos(\theta/2)F(4t)}{2\omega_a\sin^2(\theta/2)}\right]^2\Big/\left[1+\left(\frac{2\omega_a}{m}\right)\sin^2\left(\frac{\theta}{2}\right)\right] \tag{43}$$

Returning to (31), one can consider the case also when the recoil of the target can be neglected. In this limit, one finds

$$\frac{d\sigma}{d\Omega_{ac}}(s_a,s_c)=\left(\frac{d\sigma}{d\Omega_{ac}}\right)_{\text{Mott}}F^2(-|\mathbf{q}|^2)(1+\xi_c^f\cdot\xi_c)/2 \tag{44}$$

with ξ_c^f given by (37) and with the Mott cross section given by

$$\left(\frac{d\sigma}{d\Omega_{ac}}\right)_{\text{Mott}}=\left(\frac{d\sigma}{d\Omega_{ac}}\right)_{\text{Ruth.}}\left[1-v^2\sin^2(\theta/2)\right] \tag{45}$$

Here v denotes the electron's velocity, and the Rutherford cross section is

$$\left(\frac{d\sigma}{d\Omega_{ac}}\right)_{\text{Ruth.}}=\left(\frac{\alpha\omega_a}{2|\mathbf{a}|^2\sin^2(\theta/2)}\right)^2 \tag{46}$$

Finally, in the nonrelativistic limit when $\omega_a\approx 1$, one finds

$$\frac{d\sigma}{d\Omega_{\text{lab}}}(s_a,s_c)=\left(\frac{d\sigma}{d\Omega_{\text{lab}}}\right)_{\text{Ruth.}}\frac{1+\xi_c^f\cdot\xi_c}{2} \tag{47}$$

with ξ_c^f given by (39).

Of particular interest are those effects associated with a detection of a change in the direction of the polarization of the electron. Various ratios may be found from (31) which are independent of the form factor $F(4t)$. For example, with reversal of the polarization vector associated with the detector one finds

$$\frac{d\sigma(\xi_a,-\xi_c)/dt}{d\sigma(\xi_a,\xi_c)/dt}=\frac{1-\xi_c^f\cdot\xi_c}{1+\xi_c^f\cdot\xi_c} \tag{48a}$$

For the case of helical electrons with $\lambda_a=1$ and $\lambda_c=+1$ or -1, one finds

$$\frac{d\sigma(+,-)/dt}{d\sigma(+,+)/dt}=\frac{-16t(4s+m^2-1)^2}{2f(s,1,m)A(s,t)_0+16t(4s+m^2-1)^2} \tag{48b}$$

In the high-energy limit with $s \gg m$, one finds that this effect decreases with energy as

$$\frac{d\sigma(+,-)/dt}{d\sigma(+,+)/dt} \approx \frac{-t}{4s^2} \approx \frac{\sin^2(\theta/2)}{m^2\omega_a\left[1+(2\omega_a/m)\sin^2(\theta/2)\right]} \qquad (48c)$$

In the case when one neglects effects due to the recoil of the target, the ratio (48) becomes

$$\frac{d\sigma(+,-)/dt}{d\sigma(+,+)/dt} \approx \frac{\tan^2(\theta/2)}{\omega_a^{\,2}} \qquad (49a)$$

Interesting phenomena are found also when the polarization vector of the electron is initially perpendicular to the beam direction. For example, if $\xi_{a1}=1$ and $\lambda_c=+1$ or -1, then

$$\frac{d\sigma(\xi_{a1}=1,\lambda_c=-1)/dt}{d\sigma(\xi_{a1}=1,\lambda_c=1)/dt} \approx \left[\frac{1-\omega_a\cot(\theta/2)}{1+\omega_a\cot(\theta/2)}\right]^2 \qquad (49b)$$

In conclusion, one notes that the ratio in the case when $\xi_{a1}=1$ and $\xi_{c1}=+1$ or -1 is the same as (49a) and that the differential cross section vanishes when $\xi_{a2}=1$ and $\xi_{c2}=-1$. Numerical values for the ratio (48b) are given in Table I for the elastic scattering of helical electrons from helium in the laboratory system.

4. DEEP INELASTIC SCATTERING AND VIRTUAL PHOTON CROSS SECTIONS

The scattering of a high-energy electron in the state $|a, s_a\rangle$ to the final state $|c, s_c\rangle$ from a nucleon in the state $|b, s_b\rangle$ to produce a final state with N undetected hadrons characterized by total four-momentum $D=a+b-c$, may be also analyzed with the invariant cross section (11). Before developing various specific cross sections for the analysis of experimental situations involving all possible polarization configurations for the particles in the initial state and for the scattered electron, it is useful to describe the physics of this process in terms of the absorption by the target of a virtual photon of effective mass $q^2=4t$. This absorption process may be analyzed as well with the invariant cross section (11) which can be used to define the cross sections for the absorption of a virtual photon of specific polarization. Specific cross sections of this type have proven useful to other researchers

TABLE 1. The Polarization Parameter $P(s,t)=[d\sigma(+,-)/dt]/[d\sigma(+,+)/dt]$ Forms Eq. (48b) for Values of ν_0, where $\nu=\omega_a-\omega_c=\nu_0\omega_a$ for the Elastic Scattering of Helical Electrons from Helium. The Electron's Energy in Natural Units in the Laboratory System is ω_a, and the Maximum Energy loss is $\nu_{max}=f(s,1,m)/8ms$.

$\nu_0\times10^{-x}$	$P(s,t)$ $\omega_a=1.1$ $x=5$	$P(s,t)$ $\omega_a=1.01$ $x=6$	$P(s,t)$ $\omega_a=1.001$ $x=7$	$P(s,t)$ $\omega_a=1.0001$ $x=8$
0.5	0.087	0.099	0.100	0.100
1.0	0.195	0.220	0.223	0.223
1.5	0.332	0.372	0.376	0.377
2.0	0.512	0.568	0.574	0.575
2.5	0.756	0.830	0.838	0.839
3.0	1.111	1.199	1.208	1.209
3.5	1.671	1.756	1.766	1.767
4.0	2.684	2.698	2.701	2.701
4.5	5.084	4.626	4.592	4.589
5.0	17.853	10.806	10.440	10.405

$\nu_0\times10^{-x}$	$P(s,t)\times10^{-2}$ $\omega_a=5$ $x=3$	$P(s,t)\times10^{-2}$ $\omega_a=10$ $x=3$	$P(s,t)\times10^{-4}$ $\omega_a=100$ $x=2$	$P(s,t)\times10^{-6}$ $\omega_a=1000$ $x=1$
0.2	0.718	0.080	0.081	0.104
0.4	1.751	0.173	0.176	0.232
0.6	3.362	0.285	0.290	0.392
0.8	6.226	0.420	0.428	0.601
1.0	12.741	0.586	0.600	0.881
1.2	42.129	0.797	0.818	1.280
1.4		1.072	1.104	1.891
1.6		1.446	1.499	2.944
1.8		1.986	2.075	5.198
2.0		2.832	2.996	13.405
2.2		4.345	4.704	
2.4		7.863	8.966	
2.6		24.462	38.402	

for the interpretation of experimental information regarding inelastic scattering (Gilman, 1972; Hand, 1963; Bjorken, 1970).

The method which I describe here for defining the particular photon absorption cross sections is essentially a more conventional version of the method used by Schwinger (1975a, 1975b). To begin, one considers the interaction of an electron and the target to be mediated by a spacelike photon of momentum $q=a-c$ which is associated with an electromagnetic field characterized in momentum space by the vector potential $A^{\mu}(q)$. This vector potential satisfies the Lorentz condition $q\cdot A(a)=0$ and takes the form

$$A^{\mu}(q) = -[\pi/(t+i\varepsilon)]\, g^{\mu\nu} J_{\nu}(c,a)_e \tag{50a}$$

It may also be written as

$$A^{\mu}(q) = -(4\pi)^{1/2} \sum_{\lambda=1}^{3} \varepsilon^{\mu}(\lambda) A(\lambda) \tag{50b}$$

where $\varepsilon(\lambda)$ denotes the polarization four-vector for a state of pure polarization. One can choose three independent states of polarization which are compatible with the Lorentz condition such that $q \cdot \varepsilon(\lambda) = 0$. Returning to (11) with the initial virtual photon and nucleon state represented by $|q, \varepsilon(\lambda), b, s_b\rangle$ and with the transition amplitude to the final state containing N hadrons represented by

$$\langle f|M|q, \varepsilon(\lambda), b, s_b\rangle = -\alpha^{1/2} A(q) \cdot \langle b, s_b|J_h|N\rangle \tag{51}$$

one finds

$$\sigma(\varepsilon(\lambda), s_b) = \frac{\alpha 8\pi \varepsilon^{*}(\lambda) \cdot \mathrm{Im} H(q, b, s_b) \cdot \varepsilon(\lambda)}{m(\nu^2 - 4t)^{1/2}} \tag{52a}$$

For this cross section, I have used (3d) in the form

$$f(s,q,b) = 4m^2(\nu^2 - 4t)$$

and I have used the variable $\nu = (b \cdot q)/m$ which corresponds to the electron's energy loss $\omega_a - \omega_c$ in the rest frame of the target; furthermore I have introduced the definition

$$\begin{aligned} \mathrm{Im}\, H^{\mu\nu}(q, b, s_b) = \sum_{N} \sum_{\text{spin}} \frac{1}{8(2\pi)^{3N-4}} \int dp_1\, dp_2 \ldots dp_N \\ \times \prod_{i=1}^{N} \delta(p_i^2 - m_i^2)\theta(p_i)\delta(q + b - D_N) \\ \times \langle b, s_b|J_h^{\mu}|N\rangle\langle N|J_h^{\nu}|b, s_b\rangle \end{aligned} \tag{52b}$$

As can be seen from the unitarity condition for the case of the forward scattering of a virtual photon by a nucleon

$$2\,\mathrm{Im}\langle q, b|T|q, b\rangle = (2\pi)^4 \sum_{N} \delta(q + b - D_N) |\langle q, b|T|N\rangle|^2 \tag{53}$$

with

$$\langle f|T|i\rangle = \prod_i (2\mathscr{E}_i)^{-1/2} \prod_f (2\mathscr{E}_f)^{-1/2} \langle f|M|i\rangle$$

The tensor $H^{\mu\nu}(q, b, s_b)$ is proportional to the amplitude for the forward scattering of a virtual photon by the target. It follows from the gauge invariance of the scattering matrix that $H^{\mu\nu}(q, b, s_b)$ may be expressed in terms of gauge-invariant tensors such that

$$H^{\mu\nu}(q, b, s_b) = \sum_{i=1}^{4} \operatorname{Im} H(4t, \nu)_i T_i^{\mu\nu} \tag{54a}$$

with

$$T_1^{\mu\nu} = -m^2 4t(g^{\mu\nu} - q^\mu q^\nu/4t) \tag{54b}$$

$$T_2^{\mu\nu} = -4t[b^\mu - (m\nu/4t)q^\mu][b^\nu - (m\nu/4t)q^\nu]$$
$$-(1-\nu^2/4t)T_1^{\mu\nu} \tag{54c}$$

$$T_3^{\mu\nu}(s_b) = -2m^3 iE(\mu, \nu, q, s_b) \tag{54d}$$

$$T_4^{\mu\nu}(s_b) = -mq\cdot s_b iE(\mu, \nu, q, b) \tag{54e}$$

where one uses the notation

$$E(a, b, c, d) = \varepsilon^{\mu\nu\lambda\sigma} a_\mu b_\nu c_\lambda d_\sigma \tag{55a}$$

with the completely antisymmetrical pseudotensor defined according to

$$i\varepsilon^{\mu\nu\lambda\sigma} = (1/4)\operatorname{Tr}[\gamma^5\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\sigma] \tag{55b}$$

The tensors (54) may be derived from either a gauge-invariant representation of the forward photon scattering transition probability made up of scalars formed from the electromagnetic field $F^{\mu\nu} = q^\mu A^\nu(q) - q^\nu A^\mu(q)$ and the four-vectors q, b, and s_b as done in Schwinger (1975a, b) or by a direct reduction of the polarization tensor of the nucleon occurring in elastic scattering as described in Section 5. The tensors in (54) have the convenient properties

$$q_\mu T_i^{\mu\nu} = 0, \qquad i = 1, 2, 3, \text{ or } 4$$
$$b_\mu T_2^{\mu\nu} = b_\mu T_4^{\mu\nu}(s_b) = 0 \tag{56}$$

where $T_1^{\mu\nu}$ and $T_2^{\mu\nu}$ are symmetrical in μ and ν, while $T_3^{\mu\nu}(s_b)$ and $T_4^{\mu\nu}(s_b)$ are antisymmetrical.

With particular choices for the photon polarization four-vectors in (52a), one can construct as many virtual photon absorption cross sections as there are form factors $\operatorname{Im} H_i(4t,\nu)$, and these form factors may then be expressed in terms of the cross sections. The photon polarization four-vectors can be represented in a coordinate system formed from the right-handed orthonormal triad $\mathbf{e}_p$, $\mathbf{e}_q$, and $\mathbf{e}_r$, where $\mathbf{e}_q$ is in the direction of the virtual photon's momentum. This system is defined as follows:

$$\mathbf{e}_p = \mathbf{e}_r \times \mathbf{e}_q$$
$$\mathbf{e}_q = (\mathbf{a}-\mathbf{c})/|\mathbf{a}-\mathbf{c}|$$
$$\mathbf{e}_r = \mathbf{e}_a \times \mathbf{e}_c/\sin\theta_{ac} \tag{57}$$

A state of transverse polarization is represented by a unit vector $\mathbf{e}_T$ in the plane $(\mathbf{e}_p, \mathbf{e}_r)$, and the associated polarization four-vector satisfies the conditions $\varepsilon^*(T)\cdot\varepsilon(T) = -1$, and $q\cdot\varepsilon(T) = 0$. For convenience, the spacelike four-vectors $\varepsilon(p)$, and $\varepsilon(r)$ are represented in terms of four-vectors for states of circular polarization such that

$$\sqrt{2}\,\varepsilon(r) = \varepsilon(+) + \varepsilon(-)$$
$$i\sqrt{2}\,\varepsilon(p) = \varepsilon(-) - \varepsilon(+) \tag{58}$$

The cross-section for the absorption of a transverse virtual photon is found with the choice $\varepsilon(T)$ in (52) to be

$$\sigma_T = Q_0\left[(1-\nu^2/4t)\operatorname{Im} H_2(4t,\nu) - \operatorname{Im} H_1(4t,\nu)\right] \tag{59a}$$

with

$$Q_0 = -32\pi\alpha m t/\nu(1-4t/\nu^2)^{1/2} \tag{59b}$$

A state of longitudinal polarization is represented by the timelike unit four-vector

$$\varepsilon^\mu(L) = (m\nu q^\mu - 4tb^\mu)/m\left[(4t)^2 - 4t\nu^2\right]^{1/2} \tag{60}$$

When this four-vector is substituted into (52a), one finds the cross section

$$\sigma_L = Q_0 \operatorname{Im} H_1(4t,\nu) \tag{61}$$

At this point it is worthwhile to observe that the form factors $\mathrm{Im}\, H_1(4t,\nu)$, and $\mathrm{Im}\, H_2(4t,\nu)$ are related to those used by other authors (Gilman, 1972, pp. 130–133) by the equations

$$\mathrm{Im}\, H_1(4t,\nu) = -(\pi/8mt)[-W_1(4t,\nu) + (1-\nu^2/4t)W_2(4t,\nu)]$$

$$\mathrm{Im}\, H_2(4t,\nu) = -(\pi/8mt)W_2(4t,\nu) \tag{62}$$

Furthermore, the cross sections (59), and (61) differ from those used by others in the definition of the flux where the replacement $m\nu(1-4t/\nu^2)^{1/2} \to mK$ is made with $mK = m\nu + 2t$.

The remaining two form factors also may be related to cross sections for particular polarizations. One relation is obtained if the direction of the nucleon's spin vector is chosen in its rest system to be in the direction $\mathbf{e}_q$, the direction of the virtual photon's helicity. In this case the polarization four-vector for the nucleon may be written as

$$s_b^\mu = [q^\mu - (\nu/m)b^\mu]/(\nu^2 - 4t)^{1/2} \tag{63}$$

With this choice, the tensors $T_3^{\mu\nu}(s_b)$ and $T_4^{\mu\nu}(s_b)$ are related according to

$$(\nu^2 - 4t)T_3^{\mu\nu}(s_b) = 2\nu m T_4^{\mu\nu}(s_b) \tag{64}$$

Now if one uses $\varepsilon(+)$, and $\varepsilon(-)$ in (52a), it is found that

$$\sigma(\lambda_q) = \sigma_T - \lambda_q Q_0[(1-\nu^2/4t)\mathrm{Im}\, H_4(4t,\nu) - (2m\nu/4t)\mathrm{Im}\, H_3(4t,\nu)] \tag{65a}$$

$$\sigma(\uparrow\uparrow) + \sigma(\uparrow\downarrow) = 2\sigma_T \tag{65b}$$

$$\sigma(\uparrow\uparrow) - \sigma(\uparrow\downarrow) = \frac{-16\pi\alpha m}{(\nu^2-4t)^{1/2}}[2m\nu\,\mathrm{Im}\, H_3(4t,\nu) + (\nu^2-4t)\mathrm{Im}\, H_4(4t,\nu)] \tag{65c}$$

In the above $\uparrow\uparrow$ denotes parallel alignment of the nucleon's spin direction and the photon's helicity and $\uparrow\downarrow$ denotes antiparallel alignment. To complete the description, a final cross section may be found from the four-vector $\varepsilon(L)$ and the space like unit four-vector

$$\varepsilon^\mu(T_b) = E(\mu, q, b, s_b)/D^{1/2}(q, b, s_b) \tag{66}$$

with

$$D(q,b,s_b)=\begin{vmatrix} 4t & m\nu & q\cdot s_b \\ m\nu & m^2 & 0 \\ q\cdot s_b & 0 & s_b^2 \end{vmatrix}$$

It is convenient to choose s_b so that in the rest frame of the nucleon it is equal to $\pm\mathbf{e}_{q\perp}$ so that $q\cdot s_b=0$, and $s_b^2=-1$. In this frame $\varepsilon(T_b)$ becomes $(0,\pm\mathbf{e}_q\times\mathbf{e}_{q\perp})$ and $D(q,b,s_b)=m^2(\nu^2-4t)$. The desired cross section is found with the choice

$$\varepsilon(LT)=a\varepsilon(L)-ib\varepsilon(T_b) \tag{67}$$

where a and b are real constants and where $\varepsilon(LT)$ is normalized according to

$$\varepsilon^*(LT)\cdot\varepsilon(LT)=a^2-b^2$$

It now follows from (52a) that

$$\sigma(\xi_{b\perp}=\pm1)_{LT}=a^2\sigma_L+b^2\sigma_T$$
$$\pm4abQ_0\left[m/(-4t)^{1/2}\right]\mathrm{Im}\,H_3(4t,\nu) \tag{68}$$

Before concluding this discussion, it is interesting to note the results for real photons which are found when $4t=0$. It can be seen from the ratio $\sigma_L/(\sigma_L+\sigma_T)$ that σ_L vanishes and that σ_T becomes the total cross section for the absorption of photons by an unpolarized target

$$\sigma_\gamma=\sigma_T(4t=0)=8\pi\alpha m\nu\,\mathrm{Im}\,H_2(0,\nu) \tag{69a}$$

Also in this case one observes that (65c) becomes

$$\sigma_\gamma(\uparrow\uparrow)-\sigma_\gamma(\uparrow\downarrow)=-16\pi\alpha m\left[2m\,\mathrm{Im}\,H_3(0,\nu)+\nu\,\mathrm{Im}\,H_4(0,\nu)\right] \tag{69b}$$

Returning to the principal task of deriving cross sections for the analysis of experimental situations with general polarization, one can use the transition amplitude

$$\langle f|M|a,s_a,b,s_b\rangle=\alpha\langle N|J_h|b,s_b\rangle\cdot D(t)\cdot J_e(a,c) \tag{70}$$

where the photon propagator is found in (22), along with (11) to write

$$\sigma(s,t,u,s_a,s_b,s_c) = \left[\frac{\alpha^2}{2\pi f^{1/2}(s,1,m)}\right] \times \int \frac{dc\,\theta(c)\delta(c^2-1)L^{\mu\nu}(s_a,s_c)\mathrm{Im}\,H_{\mu\nu}(q,b,s_b)}{t^2+i\varepsilon} \tag{71}$$

Here $L^{\mu\nu}(s_a, s_c)$ is given by (24), and $\mathrm{Im}\,H_{\mu\nu}(q,b,s_b)$ by (52b) or (54a). As before s and t are given by (2), but u is now

$$4u = 2(1+m^2+m\nu)-4s \tag{72}$$

Using a definition similar to (15), one can define the double differential cross section

$$\frac{\partial^2\sigma}{\partial t\,\partial u}(s_a,s_b,s_c) = \sigma(s,t,u,s_a,s_b,s_c)\delta\left(t-\frac{(a-c)^2}{4}\right)\delta\left(u-\frac{(c-b)^2}{4}\right) \tag{73}$$

In the laboratory system, one finds

$$\frac{\partial^2\sigma}{\partial\Omega_{ac}\partial\omega_c}(s_a,s_b,s_c) = \frac{f^{1/2}(s,1,m)f^{1/2}(u,1,m)}{32\pi m}\frac{\partial^2\sigma(s_a,s_b,s_c)}{\partial t\,\partial u} \tag{74}$$

It now follows from (71) and (74) when the integrations are carried out in the laboratory system that

$$\frac{\partial^2\sigma}{\partial\Omega_{ac}\partial\omega_c}(s_a,s_b,s_c) = \frac{\alpha^2}{8\pi m t^2}\left[\frac{f^{1/2}(u,1,m)}{f^{1/2}(s,1,m)}\right] \times \theta(\omega_c)L^{\mu\nu}(s_a,s_c)\mathrm{Im}\,H_{\mu\nu}(q,b,s_b) \tag{75}$$

Upon evaluation of the tensor products which appear in the above, one finds

$$L^{\mu\nu}(s_a,s_c)\cdot\mathrm{Im}\,H_{\mu\nu}(q,b,s_b) = \sum_{i=1}^{4}\mathrm{Im}\,H_i(4t,\nu)L(s_a,s_c)\cdot T_i \tag{76a}$$

with

$$L(s_a, s_c)\cdot T_1 = -8m^2t[2t+1-s_a\cdot s_c] \tag{76b}$$

$$\begin{aligned} L(s_a, s_c)\cdot T_2 = -\Big\{16t\big[s(2s-1-m^2-m\nu) \\ + m\nu(m^2+1)/4+(m^2-1)^2/8\big]\cdot(1-s_a\cdot s_c) \\ +2(m\nu)^2(2t+1-s_a\cdot s_c) \\ +16t\big(s-(m^2+1)/4\big)(b\cdot s_a a\cdot s_c + c\cdot s_a b\cdot s_c) \\ -8tm\nu b\cdot s_a a\cdot s_c - 4tm^2 c\cdot s_a a\cdot s_c \\ +16t^2 b\cdot s_a b\cdot s_c - 8m^2t^2(1+s_a\cdot s_c)\Big\} \end{aligned} \tag{76c}$$

$$\begin{aligned} L(s_a, s_c)\cdot T_3(s_b) = 4m^3\big[4t(s_a\cdot s_b + s_b\cdot s_c) \\ +(q\cdot s_b)(c\cdot s_a - a\cdot s_c)\big] \end{aligned} \tag{76d}$$

$$\begin{aligned} L(s_a, s_c)\cdot T_4(s_b) = 2mq\cdot s_b\big[m\nu(c\cdot s_a - a\cdot s_c) \\ +4t(b\cdot s_a + b\cdot s_c)\big] \end{aligned} \tag{76e}$$

Results in deep inelastic scattering for all polarization orientations of the electron as well as the initial nucleon may be obtained from (75) and (76). From these equations, one may also derive results for those phenomena which are dependent upon the electron's mass and which are associated with a change in the orientation of the electron's polarization vector. Furthermore, polarization cross sections describing the elastic scattering of polarized electrons from polarized nucleons, and muons, and from spin-zero targets may be obtained directly from (76) as well.

Toward this end, I begin with a discussion of the scattering in the relativistic limit of helical electrons from a nucleon of polarization s_b. In this limit one may use the approximations $s_a \approx \lambda_a a$, $s_c \approx \lambda_c c$, and $a\cdot c = 1-2t$ in (76) to find

$$L(s_a, s_c)\cdot T_1 \approx -16m^2t^2(1+\lambda_a\lambda_c) \tag{77a}$$

$$\begin{aligned} L(s_a, s_c)\cdot T_2 \approx -32t\big[s(s-m^2/2-m\nu/2) \\ + m^2\nu(m+\nu)/8 - m^2t/4 + m^4/16\big](1+\lambda_a\lambda_c) \end{aligned} \tag{77b}$$

$$L(s_a, s_c)\cdot T_3(s_b) \approx 8m^3t\big[(a+c)\cdot s_b\big](\lambda_a+\lambda_c) \tag{77c}$$

$$L(s_a, s_c)\cdot T_4(s_b) \approx -4mtq\cdot s_b[m^2-4s+m\nu](\lambda_a+\lambda_c) \tag{77d}$$

With the approximations (41) and the definitions (62) as well as

$$d(4t,\nu) = -16m^3 \operatorname{Im} H_3(4t,\nu) \tag{78a}$$

$$g(4t,\nu) = -8m \operatorname{Im} H_4(4t,\nu) \tag{78b}$$

one finds from (75) and (77)

$$\begin{aligned}\frac{\partial^2\sigma(\lambda_a, s_b, \lambda_c)}{\partial\Omega_{ac}\partial\omega_c} \approx \frac{-\alpha^2\omega_c\theta(\omega_c)}{8t\omega_a} \\ \times\left[2W_1(4t,\nu)+\cot^2(\theta/2)W_2 4t,\nu\right](1+\lambda_a\lambda_c) \\ -\frac{\alpha^2\omega_c\theta(\omega_c)}{16\pi m t\omega_a}\left[d(4t,\nu)(a+c)\cdot s_b\right. \\ \left.+g(4t,\nu)m(\omega_a+\omega_c)q\cdot s_b\right](\lambda_a+\lambda_c)\end{aligned} \tag{79}$$

Various special cases may be obtained from (79) with particular choices of s_b. For example, with summation on the spin directions of the scattered electron and with $s_b = (0, \pm \mathbf{e}_a)$, and $\lambda_a = 1$, one finds the difference for parallel and antiparallel alignment of the electron and nucleon spin directions

$$\begin{aligned}\frac{\partial^2\sigma(\uparrow\uparrow)}{\partial\Omega_{ac}\partial\omega_c} - \frac{\partial^2\sigma(\uparrow\downarrow)}{\partial\Omega_{ac}\partial\omega_c} = \frac{\alpha^2}{4\pi m t}\frac{\omega_c}{\omega_a}\theta(\omega_c) \\ \times\left[d(4t,\nu)(\omega_a+\omega_c\cos\theta)\right. \\ \left.+g(4t,\nu)m(\omega_a+\omega_c)(\omega_a-\omega_c\cos\theta)\right]\end{aligned} \tag{80}$$

The cross-section difference resulting when the nucleon spin is parallel or antiparallel to the virtual photon direction $\mathbf{e}_q$ is found from (63) and (79) with $\lambda_a = 1$ and when the polarization of the scattered electron is undetected to be

$$\begin{aligned}\frac{\partial^2\sigma(\uparrow\uparrow)_\gamma}{\partial\Omega_{ac}\partial\omega_c} - \frac{\partial^2\sigma(\uparrow\downarrow)_\gamma}{\partial\Omega_{ac}\partial\omega_c} = \frac{\alpha^2}{4\pi m t}\frac{\omega_c}{\omega_a}(\nu^2-4t)^{-1/2}\theta(\omega_c) \\ \times\left[d(4t,\nu)(\omega_a^2-\omega_c^2)\right. \\ \left.+g(4t,\nu)m(\omega_a+\omega_c)(\nu^2-4t)\right]\end{aligned} \tag{81}$$

Another interesting cross section is obtained when a helical electron with

$\lambda_a = 1$ is scattered from a nucleon at rest which is polarized perpendicularly to the electron beam such that $s_b = (0, \xi_{b\perp}, \mathbf{e}_{a\perp})$. When the polarization of the scattered electron is undetected, one finds

$$\frac{\partial^2\sigma(\xi_{b\perp})}{\partial\Omega_{ac}\partial\omega_c} = \frac{\alpha^2}{8\pi mt}\frac{\omega_c^2}{\omega_a}\theta(\omega_c)$$
$$\times \sin\theta\cos\alpha\left[d(4t,\nu) - g(4t,\nu)m(\omega_a + \omega_c)\right] \tag{82}$$

Interesting phenomena associated with a change in the orientation of the polarization of the electron may be found from (75) and (76) as well. As demonstrated in Appendix B where the lepton polarization tensor is represented as the sum of two symmetrical gauge-invariant tensors and one antisymmetrical gauge-invariant tensor, the cross section for a change from perpendicular to parallel polarization is a first-order effect in the electron's mass, whereas a reversal of the electron's helicity is a second-order effect. Since these effects decrease with increasing s, they are more likely to be detected in the case of elastic scattering, which is described in the next section. Before turning to this case, I present here the result for deep inelastic scattering for the case when an electron initially polarized perpendicularly to the beam direction is detected with helicity $\lambda_c = \pm 1$ after being scattered from an unpolarized nucleon. In this case, one finds from (76) and (77) in the laboratory system

$$\frac{\partial^2\sigma(+)}{\partial\Omega_{ac}\partial\omega_c} - \frac{\partial^2\sigma(-)}{\partial\Omega_{ac}\partial\omega_c}$$
$$= \frac{\alpha^2}{4\pi mt^2}\left[\frac{f(u,1,m)}{f(s,1,m)}\right]^{1/2}\theta(\omega_c)$$
$$\times\Bigg[\mathrm{Im}\,H_1(4t,\nu)8m^2t(-\omega_c) + \mathrm{Im}\,H_2(4t,\nu)$$
$$\times\left(16t\left\{s\left[\omega_c m - \frac{(1+m^2)}{2}\right] + \frac{m\nu(m^2+1)}{4}\right.\right.$$
$$\left.\left.+\frac{(m^2-1)^2}{8}\right\} + 2(m\nu)^2 + 8m^2t^2\right)(-\omega_c)$$
$$+\mathrm{Im}\,H_2(4t,\nu)4tm^2\left[\omega_a(\omega_c^2 - 1) + |\mathbf{a}|\,|\mathbf{c}|\omega_c\cos\theta_{ac}\right]\Bigg]$$
$$\times\cos\alpha\sin\theta_{ac} \tag{83}$$

In addition to the results which have already been given for deep inelastic scattering, it is also possible to use the formulation developed in this section to derive the sum rule which has been given in Bjorken (1970). This rule relates the structure function $W_2(4t,\nu)$ to the asymmetry parameter for deep inelastic scattering when the helicity of the virtual photon is either parallel or antiparallel to the spin direction of the nucleon. With A and P used to denote, respectively, the antiparallel and parallel alignment of the virtual photon's helicity and the nucleon's spin, one can use (59), (61), (62), and (65a) to confirm the identity

$$\frac{\sigma_P-\sigma_A}{\sigma_T+\sigma_L}W_2(4t,\nu)=\left(\frac{2}{\pi m}\right)\left(1-\frac{\nu^2}{4t}\right)^{-1}\times\left[\varepsilon^*(P)\cdot\operatorname{Im}H(q,b)\cdot\varepsilon(P)-\varepsilon^*(A)\cdot\operatorname{Im}H(q,b)\cdot\varepsilon(A)\right] \tag{84}$$

where

$$2\sigma_T=\sigma_P+\sigma_A$$

One can also use (75) and a method similar to that used to derive (81) to show that the polarization asymmetry parameter

$$\Delta=\frac{(\partial^2\sigma_A/\partial t\,\partial u)-(\partial^2\sigma_P/\partial t\,\partial u)}{\partial^2\sigma_A/\partial t\,\partial u+\partial^2\sigma_P/\partial t\,\partial u} \tag{85a}$$

becomes

$$\Delta=(\nu^2-4t)^{1/2}\left(\frac{\omega_a+\omega_c}{2\omega_a\omega_c}\right)\left(\frac{\sigma_A-\sigma_P}{\sigma_A+\sigma_P+2\sigma_L}\right)\times\left[1+\frac{t}{\omega_a\omega_c}+\frac{\nu^2-4t}{2\omega_a\omega_c}\left(\frac{\sigma_T}{\sigma_T+\sigma_L}\right)\right]^{-1} \tag{85b}$$

For the case of interest where $\omega_a\gg\omega_c$ and where $\nu^2>-4t$, the asymmetry parameter becomes

$$\Delta\approx\frac{\sigma_A-\sigma_P}{\sigma_A+\sigma_P+2\sigma_L} \tag{86}$$

In the scaling limit where $W_2(4t, \nu)$ becomes a function of a single variable, equation (84) can be used to show that

$$\lim_{-4t \to \infty} \int_0^\infty \left(\frac{\sigma_A - \sigma_P}{\sigma_T + \sigma_L} \right) W_2(4t, \nu)\, d\nu = Z \tag{87}$$

where Z is a constant. As described in Bjorken (1970), the right-hand side of (79) may be converted, with the aid of an off-shell δ function, into a relation involving integration over the commutator of the components J_x and J_y of the hadronic current. With the use of the quark model current algebra, the resulting expression may be evaluated. In this way, one can achieve an estimate of the magnitude of the polarization asymmetry parameter.

5. ELASTIC SCATTERING OF POLARIZED ELECTRONS ON POLARIZED MUONS AND POLARIZED NUCLEONS

Information regarding the polarization effects associated with the elastic scattering of a polarized electron from a polarized nucleon may be found directly from (71) and (76). When the tensors (54) are supplemented with additional gauge-invariant tensors, one can also derive information related to the polarization of the target after scattering. To demonstrate how the method works, I begin with a description of the elastic scattering of an electron from a spin-zero target. Returning to (71) and integrating over u to form

$$\sigma(s, t, s_a, s_c) = \int \sigma(s, t, u, s_a, s_c) \delta\left(4u - (c-b)^2\right) d4u \tag{88}$$

one finds the invariant cross section

$$\sigma(s, t, s_a, s_c) = \left(\alpha^2/2\pi f^{1/2}(s, 1, m)\right) \iint dc\, d2m\nu\theta(c)$$

$$\times\, \delta(c^2 - 1) L^{\mu\nu}(s_a, s_c) \mathrm{Im}\, H_{\mu\nu}(q, b)/(t^2 + i\varepsilon) \tag{89}$$

Now with

$$(4/\pi)\mathrm{Im}\, H_1(4t, \nu) = -(1/t)(1 - \nu^2/4t) F^2(4t)\theta(d)\delta(4t + 2m\nu)$$

$$(4/\pi)\mathrm{Im}\, H_2(4t, \nu) = -(1/t) F^2(4t)\theta(d)\delta(4t + 2m\nu)$$

$$\mathrm{Im}\, H_3(4t, \nu) = \mathrm{Im}\, H_4(4t, \nu) = 0 \tag{90}$$

it follows that

$$p \cdot L(s_a, s_c) \cdot p = -(1/t)\big[(1 - t/m^2) L(s_a, s_c) \cdot T_1 + L(s_a, s_c) \cdot T_2\big] \tag{91}$$

Finally, with the use of (76), one recovers the results (30) and (32).

The elastic scattering of electrons and nucleons may be studied in a similar manner; however, for a complete analysis in which the polarization of the scattered nucleon as well as the polarization of the scattered electron is also detected, the expressions (76) must be supplemented with additional terms. To proceed, one observes that elastic scattering is described in terms of the amplitude

$$\langle c, s_c, d, s_d | M | a, s_a, b, s_b \rangle = \alpha \bar{u}(d) \Gamma u(b) \cdot D(t) \cdot J(a, c)_e \tag{92}$$

with $D(t)$ given in (22). The relevant quantity which must be calculated so as to obtain the cross sections from (11) is

$$L(s_a, s_c) \cdot M(s_b, s_d)_{\text{nuc.}} = L^{\mu\nu}(s_a, s_c) \operatorname{Tr}\big[\rho_d \Gamma_\mu \rho_b \Gamma_\nu\big] \tag{93}$$

where ρ_b and ρ_d are defined as in (26) and where

$$\Gamma^\mu = G_1 \gamma^\mu + G_2 p^\mu \tag{94}$$

The form factors G_1 and G_2 are related to the electric and the magnetic form factors according to

$$G_1 = F_m(4t), \qquad G_2 = 2m\big[F_e(4t) - F_m(4t)\big]/p^2 \tag{95}$$

with $p = 2b + q$. It is instructive to note that the nucleon polarization tensor may be expressed as the sum of gauge-invariant tensors in the form

$$M^{\mu\nu}(s_b, s_d)_{\text{nuc.}} = G_1^2 M^{\mu\nu}(s_b, s_d) + G_2^2 \operatorname{Tr}[\rho_d \rho_b] p^\mu p^\nu + G_1 G_2 \big(\operatorname{Tr}[\rho_d p^\mu \rho_b \gamma^\nu] + \operatorname{Tr}[\rho_d \gamma^\mu \rho_b p^\nu]\big) \tag{96}$$

with

$$M^{\mu\nu}(s_b, s_d) = M^{\mu\nu}(0,0) + MS^{\mu\nu}(s_b, s_d) + MA^{\mu\nu}(s_b, s_d) \tag{97}$$

where

$$M^{\mu\nu}(0,0) = \operatorname{Tr}\big[\rho_{0d} \gamma^\mu \rho_{0b} \gamma^\nu\big] \tag{98a}$$

$$MS^{\mu\nu}(s_b, s_d) = \operatorname{Tr}\big[\rho_{0d} \not{s}_d \gamma^\mu \rho_{0-b} \not{s}_b \gamma^\nu\big] \tag{98b}$$

$$MA^{\mu\nu}(s_b, s_d) = imE(\mu, \nu, q, s_b + s_d) \tag{98c}$$

The first two tensors are symmetrical in μ and ν, whereas the latter is antisymmetrical. A further reduction of (97) yields the more useful representation

$$\begin{aligned} M^{\mu\nu}(s_b, s_d)_{\text{nuc.}} = M^{\mu\nu}(0,0)_{\text{nuc.}} \\ + G_1^2\left[MS^{\mu\nu}(s_d, s_b) + MA^{\mu\nu}(s_b, s_d)\right] \\ - G_2^2\left[q\cdot s_b q\cdot s_d + 2(m^2 - t)s_b\cdot s_d\right] p^\mu p^\nu \\ + G_1 G_2\left[A^{\mu\nu}(s_b, s_d) - S^{\mu\nu}(s_b, s_d)\right] \end{aligned} \tag{99}$$

where the antisymmetrical gauge-invariant tensor $A^{\mu\nu}(s_b, s_d)$ is defined as

$$\begin{aligned} -mA^{\mu\nu}(s_b, s_d) &= im\left[p^\mu E(q, b, s_b + s_d, \nu) \right. \\ &\quad \left. - p^\nu E(q, b, s_b + s_d, \mu)\right] \\ &= (p^2/4m^2)T_3^{\mu\nu}(s_b + s_d) - T_4^{\mu\nu}(s_b - s_d) \end{aligned} \tag{100}$$

and where the symmetrical gauge invariant tensor $S^{\mu\nu}(s_b, s_d)$ is defined as

$$\begin{aligned} S^{\mu\nu}(s_b, s_d) = p^\mu \mathrm{Tr}\left[\gamma^\nu \rho_{0d} \not{s}_d \rho_{0-b} \not{s}_b\right] \\ - p^\nu \mathrm{Tr}\left[\gamma^\mu \rho_{0-b} \not{s}_b \rho_{0d} \not{s}_d\right] \end{aligned} \tag{101}$$

As in the case of scattering from a spin-zero target, the tensor product (93) which appears in (89) may be obtained when $s_d = 0$ from (76). This is accomplished with the aid of

$$L(s_a, s_c)\cdot M(s_b, 0)_{\text{nuc.}} = (4/\pi)\int d(2m\nu) L^{\mu\nu}(s_a, s_c) \mathrm{Im}\, H_{\mu\nu}(q, b, s_b) \tag{102}$$

with

$$\begin{aligned} \mathrm{Im}\, H_1(4t, \nu) &= -(\pi/8t)F_e^2 \delta(4t + 2m\nu) \\ \mathrm{Im}\, H_2(4t, \nu) &= -\frac{(\pi/8t)\left(F_e^2 - F_m^2 t/m^2\right)\delta(4t + 2m\nu)}{1 - t/m^2} \\ \mathrm{Im}\, H_3(4t, \nu) &= -(\pi/8m^2)F_e F_m \delta(4t + 2m\nu) \\ \mathrm{Im}\, H_4(4t, \nu) &= -\frac{(\pi/8m^2)F_m(F_m - F_e)\delta(4t + 2m\nu)}{(1 - t/m^2)} \end{aligned} \tag{103}$$

It can now be seen from (98) and (99) that the cross sections for the scattering arrangement with averaging over the spin directions of the target, $s_b = 0$, and with detection of the polarization of the scattered target is obtained in a similar manner; however, in this case, one must make the replacement $s_b \to -s_d$ in the coefficient of $\operatorname{Im} H_4(4t, v)$ in (76a).

For the case when the polarizations of both particles in the initial state are known and when both polarizations of the particles in the final state are detected, one must also evaluate the contributions from the tensor products $p \cdot L(s_a, s_c) \cdot p$, $L(s_a, s_c) \cdot MS(s_b, s_d)$, and $L(s_a, s_c) \cdot S(s_b, s_d)$. The first contribution can be found from (91) and is given by (32). The two remaining contributions are rather long and may be found in Appendix A.

Before proceeding to the specific examples for the various polarization configurations in the scattering of electrons and nucleons, it is worthwhile to consider the simpler case of the scattering of polarized electrons and polarized muons. This is a special case of (99), (102), and (103) with $F_m(4t)$, $= F_e(4t) = 1$ so that one must evaluate $L(s_a, s_c) \cdot M(s_b, s_d)$, where $M^{\mu\nu}(s_b, s_d)$ is given by (97). In this case the square of the scattering amplitude which appears in (20) becomes

$$\begin{aligned}\mathscr{M}(s, t, s_a, s_b, s_c, s_d) = (\alpha\pi/t)^2 [&L(0,0) \cdot M(0,0) \\ &+ LS(s_a, s_c) \cdot M(0,0) + L(0,0) \cdot MS(s_b, s_d) \\ &+ LA(s_a, s_c) \cdot MA(s_b, s_d) + LS(s_a, s_c) \cdot MS(s_b, s_d)]\end{aligned} \tag{104}$$

With the aid of (54) this may be expressed in the form

$$\begin{aligned}\mathscr{M}(s, t, s_a, s_b, s_c, s_d) = (\alpha\pi/t)^2 \{&(-1/2t)[L(s_a, s_c) \\ &\times (T_1 + T_2 + (t/m^2) T_3 (s_b + s_d))]_{m\nu = -2t} \\ &+ LS(s_a, s_c) \cdot MS(s_b, s_d)\}\end{aligned} \tag{105}$$

All but the last term may be found from (76) and (102), and this term is given in Appendix A.

Specific results for various polarization configurations may now be derived from the above. I begin here with the case when $s_b = s_d = 0$ which illustrates the effect of the scattering process on the electron's polarization vector. Although one could derive an expression for the polarization vector ξ_c^f as is done in Section 3, I give here only the result for the invariant differential cross section from which the polarization vector of the electron after scattering can be obtained. Using (6), (8), and (20), one finds

$$\frac{d\sigma}{dt}(s_a,s_c)=\left(\frac{2}{4\pi}\right)f(s,1,m)\left(\frac{\alpha\pi}{t}\right)^2$$
$$\times\Bigg(A(s,t)_{\text{muon}}(1+\lambda_a\lambda_c)+\frac{8(4s4t)}{f(s,1,m)}(t+\mathscr{E}_b^2)\lambda_a\lambda_c$$
$$+\left\{\left[A(s,t)_{\text{muon}}-8t^2\right](\cos\alpha\cos\beta\cos\varphi_{ac}+\sin\alpha\sin\beta)\right.$$
$$\left.-16t(s-1/4)\cos\alpha\cos\beta(1+\cos\varphi_{ac})\right\}\xi_{a\perp}\xi_{c\perp}$$
$$+16\cos\alpha\sin\varphi_{ac}\mathscr{E}_a\left[\frac{(4s-1-m^2)\mathscr{E}_b}{8\mathscr{E}_a}+\frac{t}{4}\right]\xi_{a\perp}\lambda_c$$
$$-16\cos\beta\sin\varphi_{ac}\mathscr{E}_a\left[\frac{(4s-1-m^2)\mathscr{E}_b}{8\mathscr{E}_a}+\frac{t}{4}\right]\xi_{c\perp}\lambda_a\Bigg) \tag{106a}$$

with $\mathscr{E}_a$, and $\mathscr{E}_b$, and φ_{ac} given by (3) and with

$$A(s,t)_{\text{muon}}=16\left\{\left[s-(m^2+1)/4\right]^2+st+t^2/2\right\} \tag{106b}$$

In the relativistic limit (106a) becomes

$$\frac{d\sigma}{dt}(s_a,s_c)\approx\left[\frac{2}{4\pi f(s,0,m)}\right]\left(\frac{\alpha\pi}{t}\right)^2A(s,t)_{\text{muon}}$$
$$\times\left\{1+\lambda_a\lambda_c+\left[1-\frac{8t^2}{A(s,t)_{\text{muon}}}\right]\cos(\alpha-\beta)\xi_{a\perp}\xi_{c\perp}\right\}_{m_e=0} \tag{106c}$$

A number of interesting special cases may be obtained for states of pure polarization when the direction of the electron's polarization is changed upon detection. Of particular interest are the two cases where in the first $\xi_{a\perp}=\lambda_c=1$, $\lambda_a=\xi_{c\perp}=0$ and where in the second case $\lambda_a=1$, $\lambda_c=-1$, and $\xi_{a\perp}=\xi_{c\perp}=0$. The results for the first case may be read off directly from (106a). The result for the second case is conveniently represented by the parameter

$$P(s,\varphi_{ac}) = \frac{d\sigma(+,-)/dt}{d\sigma(+,-)/dt + d\sigma(+,+)/dt} = \frac{-4t\left[16st + (4s+m^2-1)^2\right]}{f(s,1,m)A(s,t)_{\text{muon}}} \tag{107}$$

where in the c.m. system t is given by (34). The numerical values for $P(s,\varphi_{ac})$ are given in Figure 2.

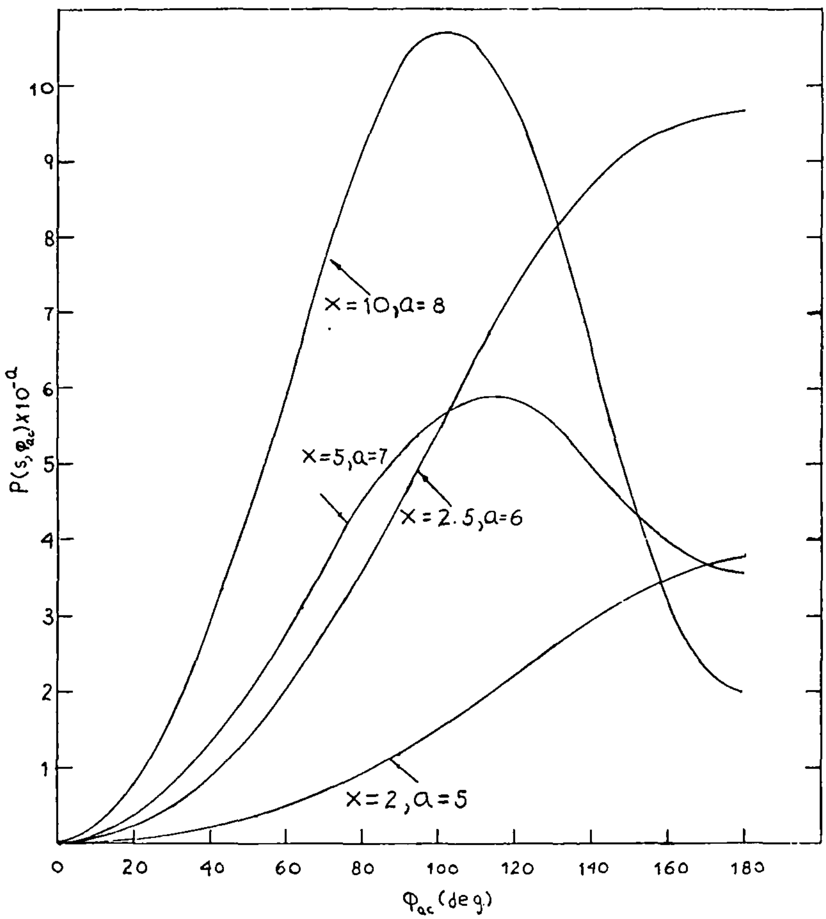

Fig. 2. The polarization parameter $P(s,\varphi_{ac})$ from Eq. (107) for the elastic scattering of helical electrons from unpolarized muons versus the scattering angle in the c.m. system φ_{ac}. The quantity x is defined by $(4s)^{1/2} = xm$.

Another interesting case occurs in the scattering of electrons and muons with known polarizations s_a, and s_b when the polarizations of the particles in the final state are undetected. In this case one finds from (105)

$$\frac{d\sigma}{dt}(s_a, s_b) = \left[\frac{4}{4\pi f(s,1,m)}\right]\left(\frac{\alpha\pi}{t}\right)^2$$
$$\times\Bigg(16\left\{\left[s-(1+m^2)4\right]^2 + st + \frac{t^2}{2}\right\}$$
$$-4t\Bigg\{\Bigg[4s-1-m^2+\frac{2t}{f(s,1,m)}$$
$$(4s-1+m^2)(4s+1-m^2)\Bigg]\lambda_a\lambda_b$$
$$+2m\left[-\cos(\alpha-\gamma)+\cos\alpha\cos\gamma\cos^2(\varphi_{ac}/2)\right]\xi_{a\perp}\xi_{b\perp}$$
$$+(\mathscr{E}_b\cos\alpha\sin\varphi_{ac})\lambda_b\xi_{a\perp} - (m\mathscr{E}_a\cos\gamma\sin\varphi_{ac})\lambda_a\xi_{b\perp}\Bigg\}\Bigg) \tag{108a}$$

where $\mathscr{E}_a$, $\mathscr{E}_b$, and φ_{ac} are given in (3). Of special interest are the cases when $\lambda_a = \xi_{b\perp} = 1$, $\lambda_b = \xi_{a\perp} = 0$ and when $\xi_{a\perp} = \lambda_b = 1$, $\xi_{b\perp} = \lambda_a = 0$. These may be found directly from (108a). The relativistic limit of this expression for the scattering of helical electrons and helical muons of parallel and antiparallel orientations yields

$$\frac{d\sigma}{dt}(\uparrow\uparrow) = \pi\left[\frac{\alpha}{t(s-m^2/4)}\right]^2\left[\left(u-\frac{m^2}{4}\right)^2 + \frac{m^2}{4}t\right] \tag{108b}$$

$$\frac{d\sigma}{dt}(\uparrow\downarrow) = \pi\left[\frac{\alpha}{t(s-m^2/4)}\right]^2\left[\left(s-\frac{m^2}{4}\right)^2 + \frac{m^2}{4}t\right] \tag{108c}$$

To conclude the discussion of the scattering of electrons and muons, one may note that the case when one averages over the spin directions of the initial particles, $s_a = s_b = 0$, and detects the polarizations of the scattered particles may be obtained from (108a) with the use of the replacements

$$\lambda_a \to -\lambda_c, \lambda_b \to -\lambda_d, \alpha \to \beta, \gamma \to \delta, \xi_{b\perp} \to \xi_{d\perp}, \text{ and } \xi_{a\perp} \to \xi_{c\perp}$$

A similar analysis may now be applied to the case of the elastic scattering of polarized electrons and polarized nucleons. Since the general method has been well illustrated, I give here only the results for some of the simpler special cases. Firstly one may recover the well-known result for the scattering of unpolarized particles when the polarizations of the particles in the final state are undetected. This result is found from (20), (76), (102), and (103) to be

$$\frac{d\sigma}{dt}(s,t)=\frac{\pi\alpha^2}{t^2 f(s,1,m)}A(s,t)_{\text{nucl.}} \tag{109a}$$

with

$$A(s,t)_{\text{nucl.}}=\frac{16}{(1-t/m^2)}\times\left\{F_e^2\left[-su-\frac{t}{4}+\frac{(m^2+1)^2}{16}\right]-F_m^2\left(\frac{t}{m^2}\right)\left[-su-\frac{tm^2}{2}\left(1-\frac{t}{m^2}\right)+\frac{(m^2-1)^2}{16}\right]\right\} \tag{109b}$$

In the relativistic limit, one finds in the laboratory system from (17), (41) and the above the Rosenbluth formula

$$\left(\frac{d\sigma(s,t)}{d\Omega_{ac}}\right)_{\text{Ros.}}=\frac{\alpha^2\cos^2(\theta/2)}{4\omega_a{}^2\sin^4(\theta/2)\left[1+(2\omega_a/m)\sin^2(\theta/2)\right]}\times\left[\frac{F_e^2-F_m^2(t/m^2)}{1-t/m^2}-\frac{2F_m^2 t}{m^2}\tan^2\left(\frac{\theta}{2}\right)\right] \tag{110}$$

The result for the general elastic scattering situation in which the initial polarizations are known and the final polarizations are detected may be found from (20), (76), (99), and (103). In this case the invariant differential cross section becomes

$$\frac{d\sigma}{dt}(s_a, s_b, s_c, s_d) = \frac{1}{4\pi f(s,1,m)}\left(\frac{\alpha\pi}{t}\right)^2$$

$$\times\left\{ - F_e^2 \frac{L(s_a, s_c)\cdot T_1}{2t} - \frac{F_e^2 - (t/m^2)F_m^2}{1 - t/m^2}\frac{L(s_a, s_c)\cdot T_2}{2t}\right.$$

$$- \frac{F_e F_m L(s_a, s_c)\cdot T_3(s_b + s_d)}{2m^2}$$

$$\frac{- F_m(F_m - F_e)L(s_a, s_c)\cdot T_4(s_b - s_d)}{(1 - t/m^2)2m^2} + G_1^2 L(s_a, s_c)\cdot MS(s_b, s_d)$$

$$- G_2^2\left[q\cdot s_b q\cdot s_d + 2(m^2 - t)s_b\cdot s_d\right] p\cdot L(s_a, s_c)\cdot p$$

$$\left. - G_1 G_2 L(s_a, s_c)\cdot S(s_b, s_d)\right\}_{m\nu = -2t} \tag{111}$$

In this expression, the tensor products $L(s_a, s_c)\cdot T_i$ come from (76), $p\cdot L(s_a, s_c)\cdot p$ is found from (91), and the remaining terms are given in Appendix A. In the relativistic limit for the scattering of a helical electron, one finds from the above and (77) with $m\nu + 2t = 0$ for the case when the final state polarization of the nucleon is undetected the result

$$\frac{d\sigma}{dt}(\lambda_a, s_b, \lambda_c) = \left[\frac{2}{4\pi f(s,0,m)}\right]\left(\frac{\alpha\pi}{t}\right)^2$$

$$\times\left\{ F_e^2 8m^2 t(1 + \lambda_a\lambda_c) + \frac{(F_e^2 - F_m^2 t/m^2)}{(1 - t/m^2)}16\right.$$

$$\times\left[\left(s - \frac{m^2}{4}\right)^2 + st - \left(\frac{m^2 t}{2}\right)\left(1 - \frac{t}{m^2}\right)\right](1 + \lambda_a\lambda_c)$$

$$- F_e F_m 4mt\left[(a + c)\cdot s_b\right](\lambda_a + \lambda_c)$$

$$\left. + \frac{F_m(F_m - F_e)(2t/m)}{(1 - t/m^2)} q\cdot s_b[2(u - s)](\lambda_a + \lambda_c)\right\} \tag{112}$$

In conclusion, one notes that cross sections related by the use of crossing symmetry to the ones derived above provide a description of analogous phenomena which occur in the scattering of polarized electrons and positrons. Furthermore, since the polarization cross sections which have been presented here depend only upon the well-established properties of the electromagnetic interaction and since they are independent of any particular speculations about the internal structure of the nucleon, they may be used to verify predictions of models which provide explicit representations of the structure functions. Especially interesting are those models which represent the structure functions in deep inelastic scattering (Kuti and Weisskopf, 1971; Domokos et al., 1971; Schwinger, 1976a,b). A discussion of these models may be found in Appendix C.

APPENDIX A: TRACES

The direct evaluation of the traces which have been used for deriving various cross sections may be accomplished in all cases considered with the methods described in Garavaglia (1975) and with the exchange operator method which I describe here. The latter method is demonstrated with the evaluation of $T(c,a)\cdot T(d,b)$, where

$$T^{\mu\nu}(d,b) = \mathrm{Tr}[\rho_{0d}\gamma^{\mu}\rho_{0b}\gamma^{\nu}]_{m=1} \tag{A1}$$

One may write

$$T^{\mu\nu}(d,b) = P(db)d^{\mu}b^{\nu} \tag{A2}$$

where the exchange operator $P(db)$ is defined as

$$P(d,b) = 1 + (12)_{db} + G(d,b) \tag{A3}$$

with

$$(12)_{db}d^{\mu}b^{\nu} = b^{\mu}d^{\nu};\ G(db)d^{\mu}b^{\nu} = g^{\mu\nu}(1 - d\cdot b) \tag{A4}$$

It now follows that

$$\begin{aligned} T(c,a)\cdot T(d,b) = [&1 + (12)_{db} + (12)_{ca} + (12)_{db}(12)_{ca} \\ &+ [1 + (12)_{ca}]G(db) + [1 + (12)_{db}]G(ca) \\ &+ G(db)G(ca)]\, d^{\mu}b^{\nu}c_{\mu}a_{\nu} \end{aligned} \tag{A5}$$

which yields

$$T(c,a)\cdot T(d,b) = 2[d\cdot cb\cdot a + b\cdot cd\cdot a + 2 - d\cdot b - c\cdot a] \qquad \text{(A6)}$$

A complete description of the polarization phenomena associated with the exchange of a single virtual photon in the elastic scattering of electrons from muons and nucleons is accomplished with the additional tensor product which is given here. The last tensor product in (105b) becomes upon evaluation

$$\begin{aligned} LS(s_a,s_c)\cdot MS(s_b,s_d) = &- s_a\cdot s_c L(0,0)\cdot MS(s_b,s_d) - s_b\cdot s_d M(0,0)\cdot LS(s_a,s_c) \\ &- s_a\cdot s_c s_b\cdot s_d L(0,0)\cdot M(0,0) + F(abcd) \qquad \text{(A7a)} \end{aligned}$$

The first three terms are found from (76), and the last term is

$$\begin{aligned} F(abcd) = \; & 2a\cdot ba\cdot s_c[b\cdot s_d s_a\cdot s_b + q\cdot s_b s_a\cdot s_d] \\ & + 2a\cdot bc\cdot s_a[b\cdot s_d s_b\cdot s_c + q\cdot s_b s_c\cdot s_d] \\ & + 8t^2[s_a\cdot s_b s_c\cdot s_d + s_a\cdot s_d s_b\cdot s_c] \\ & + 2[a\cdot s_c b\cdot s_a + c\cdot s_a b\cdot s_c] \\ & \times [a\cdot s_b b\cdot s_d + a\cdot s_d q\cdot s_b] \\ & - 2a\cdot s_c c\cdot s_a b\cdot s_d q\cdot s_b \\ & + 4t\{a\cdot s_c[a\cdot s_d s_a\cdot s_b + a\cdot s_b s_a\cdot s_d] \\ & \quad + c\cdot s_a[a\cdot s_b s_c\cdot s_d + a\cdot s_d s_b\cdot s_c] \\ & \quad + b\cdot s_d[b\cdot s_c s_a\cdot s_b + b\cdot s_a s_b\cdot s_c] \\ & \quad + q\cdot s_b[b\cdot s_a s_c\cdot s_d + b\cdot s_c s_a\cdot s_d] \\ & \quad + b\cdot s_d[a\cdot s_c s_a\cdot s_b - q\cdot s_b s_a\cdot s_c] \\ & \quad + a\cdot s_c[b\cdot s_d s_a\cdot s_b - c\cdot s_a s_b\cdot s_d]\} \qquad \text{(A7b)} \end{aligned}$$

The symmetrical tensor $S^{\mu\nu}(s_b,s_d)$ defined in (101) can be written

$$\begin{aligned} S^{\mu\nu}(s_b,s_d)/m = \; & 2p^\mu p^\nu s_b\cdot s_d \\ & - (p^\mu s_d^\nu + p^\nu s_d^\mu)q\cdot s_b + (p^\mu s_b^\nu + p^\nu s_b^\mu)q\cdot s_d \qquad \text{(A8)} \end{aligned}$$

and it satisfies $q_\mu S^{\mu\nu}(s_b, s_d) = 0$. From the lepton polarization tensor (B4), one can now find

$$L(s_a, s_c)\cdot S(s_b, s_d)/2m = p\cdot L(s_a s_c)\cdot p s_b\cdot s_d - p\cdot[L(0,0) + LS(s_a, s_c)]\cdot s_d q\cdot s_b + p\cdot[L(0,0) + LS(s_a, s_c)]\cdot s_b q\cdot s_d \qquad \text{(A9a)}$$

with $p\cdot L(s_a, s_c)\cdot p$ given by (32) and with

$$p\cdot L(0,0)\cdot s_d = 2[a\cdot b(a+c)\cdot s_d + (a+b)\cdot s_d 2t] \qquad \text{(A9b)}$$

$$p\cdot LS(s_a, s_c)\cdot s_d = -s_a\cdot s_c p\cdot L(0,0)\cdot s_d + 2a\cdot s_c[c\cdot s_d b\cdot s_a + a\cdot b s_a\cdot s_d] + 2c\cdot s_a[a\cdot s_d b\cdot s_c + a\cdot b s_c\cdot s_d] + 4t[(a+b)\cdot s_c s_a\cdot s_d + b\cdot s_a s_c\cdot s_d] - 2a\cdot s_c c\cdot s_a b\cdot s_d \qquad \text{(A9c)}$$

APPENDIX B: THE RELATIVISTIC LIMIT

Representations of cross sections which involve the lepton polarization tensor $L^{\mu\nu}(s_a, s_c)$ may be found in the relativistic limit, $m_e/s \approx 0$, with the method described in this appendix. Firstly, one observes that in this limit the polarization density matrices ρ_a and ρ_c may be reduced to a simpler form. To illustrate this, one notes that the polarization density matrix ρ_a for an electron moving in the direction $\mathbf{e}_a$ may be written

$$\rho_a = \frac{(\not{a} + m_e)}{2}(1 + \not{s}_a\gamma^5) \qquad \text{(B1)}$$

with

$$s_a = (a^3\lambda_a/m_e, a^0\lambda_a/m_e, \mathbf{s}_{a\perp}) \qquad \text{(B2a)}$$

$$\not{a} = a^+\gamma^- + a^-\gamma^+ \qquad \text{(B2b)}$$

$$\not{s}_a = s_a^+\gamma^- + s_a^-\gamma^+ - \mathbf{s}_{a\perp}\cdot\boldsymbol{\gamma}_\perp \qquad \text{(B2c)}$$

where

$$a^+ = (a^0 + a^3)/\sqrt{2}, \qquad a^- = (a^0 - a^3)/\sqrt{2} \qquad \text{(B2d)}$$

$$s_a^+ = (s_a^0 + s_a^3)/\sqrt{2}, s_a^- = (s_a^0 - s_a^3)/\sqrt{2} \qquad \text{(B2e)}$$

In the relativistic limit a^- vanishes, and since $\gamma^-\gamma^- = 0$, one finds with $\not{a} \approx a^+\gamma^-$ and $\not{s}_a \approx a^+\gamma^-\lambda_a + \not{s}_{a\perp}$ the result

$$2\rho_a = \not{a}\left[1 + (\lambda_a + \not{s}_{a\perp})\gamma^5\right] \tag{B3}$$

A similar expression may be found for ρ_c, and the representation of the lepton polarization tensor is found in this limit with the use of these approximations for the polarization density matrices. It is instructive to note that the properties of γ^5 permit one to write the lepton polarization tensor (24) as the sum of three gauge-invariant tensors

$$L^{\mu\nu}(0,0) = \mathrm{Tr}\left[\rho_{0c}\gamma^\mu\rho_{0a}\gamma^\nu\right] \tag{B4a}$$

$$LS^{\mu\nu}(s_a, s_c) = \mathrm{Tr}\left[\rho_{0c}\not{s}_c\gamma^\mu\rho_{0-a}\not{s}_a\gamma^\nu\right] \tag{B4b}$$

$$LA^{\mu\nu}(s_a, s_c) = -iE(\mu, \nu, q, s_a + s_c) \tag{B4c}$$

The first two tensors are symmetrical in μ and ν, whereas the latter is antisymmetrical. A particularly simple result is found for helical electrons where $s_a \approx \lambda_a a$, and $s_c \approx \lambda_c c$ so that

$$\begin{aligned} L^{\mu\nu}(\lambda_a, \lambda_c) = {} & \left[2(a^\mu a^\nu + g^{\mu\nu}t) - a^\mu q^\nu - a^\nu q^\mu\right](1 + \lambda_a\lambda_c) \\ & + (\lambda_a + \lambda_c)iE(\mu, \nu, a, q) \end{aligned} \tag{B5}$$

For scattering processes in which a change is detected in the orientation of the electron's polarization, one can use (B4) to show that the cross section for an electron initially in a state of perpendicular polarization to be detected after scattering in a state of parallel polarization is proportional to m_e, whereas the cross section representing a reversal in the direction of the electron's helicity is proportional to $m_e{}^2$. This is seen with the aid of the approximations

$$\not{s}_c = (\not{s}_{c\|}, \not{s}_{c\perp}) \tag{B6a}$$

$$s_{c\|} \approx (\lambda_c/m_e)\not{c} + \lambda_c m_e \not{\varepsilon} \tag{B6b}$$

$$\varepsilon = \left(-1/2c^0, \mathbf{e}_c/2|\mathbf{c}|\right) \tag{B6c}$$

For the first case when $\xi_{a\perp} = \lambda_c = 1$ one finds

$$\begin{aligned} 4\mathrm{Tr}\left[\rho_{0c}\not{s}_{c\|}\gamma^\mu\rho_{0-a}\not{s}_{a\perp}\gamma^\nu\right] \approx {} & m_e{}^2\mathrm{Tr}\left[\not{s}_{c\|}\gamma^\mu\not{s}_{a\perp}\gamma^\nu\right] \\ & - \mathrm{Tr}\left[\not{c}\not{s}_{c\|}\gamma^\mu\not{a}\not{s}_{a\perp}\gamma^\nu\right] \end{aligned} \tag{B7}$$

where both terms are proportional to m_e since $\not{c}\not{c} = m_e^2$. For the second case when $\lambda_a = -\lambda_c = 1$, one finds

$$4\,\mathrm{Tr}\left[\rho_{0c}\not{s}_{c\|}\gamma^{\mu}\rho_{0-a}\not{s}_{a\|}\right] \approx m_e^2\,\mathrm{Tr}\left[\not{s}_{c\|}\gamma^{\mu}\not{s}_{a\|}\gamma^{\nu}\right] - \mathrm{Tr}\left[\not{c}\not{s}_{c\|}\gamma^{\mu}\not{a}\not{s}_{a\|}\gamma^{\nu}\right] \tag{B8}$$

Following a similar method, one can see that in this expression there are zeroth- and second-order effects in m_e; however, as can be seen from (B5), the zeroth-order term vanishes for a reversal in the direction of the helicity.

APPENDIX C: STRUCTURE FUNCTIONS

In this appendix, I review two different approaches to the problem of calculating the structure functions which occur in deep inelastic scattering. The discussion is restricted to the case of the scattering of unpolarized particles; however, it can be readily extended to cover the more general case.

I begin with an outline of the well-known quark model which is described in detail in Kuti and Weisskopf (1971). In this model, it is assumed that a nucleon at rest consists of three fractionally charged spin-1/2 particles which carry the internal quantum numbers of $SU(3)$. When the nucleon is moving with high velocity, these valence quarks are accompanied by a collection of quark–antiquark paris. The interaction of a high-energy electron with a nucleon is interpreted as resulting from the interaction of the electron's virtual photon field with the quarks in the nucleon. Associated with each quark is a current of the form

$$j(p,\lambda)_i = e_i\bar{u}(p,\lambda)_i\gamma^{\mu}u(p,\lambda)_i \tag{C1}$$

where e_i ($i = 1$, 2, or 3) denotes the fractional charge on the three different types of quarks. It is assumed that the momentum of each quark is a fraction of the total momentum of the nucleon such that $p_i = x_i b$. The differential cross section for the scattering from a quark of index i is found from (74) and (A1) to be

$$\frac{\partial^2\sigma}{\partial t\,\partial\nu} = -\left(\frac{\alpha}{t}\right)^2\pi(\omega\nu)^{-1}\sum_{i=1}^{3} e_i^2 G_i(\omega) \tag{C2}$$

where $\omega = -(2m\nu/4t)$. In the high-energy limit one can use (75) to show that

$$\omega\nu W_2(4t,\nu) = \sum e_i^2 G_i(\omega) \tag{C3}$$

and the structure function becomes a function of the single scaling variable ω.

In the equations above, the functions $G_i(\omega)$ represent the probability distribution for the momenta of the quarks. These can be calculated if one assumes that the distribution for the longitudinal momenta of the core quarks is given by

$$dP_c(x) \simeq g\,dx\left(x^2 + m^2/b^2\right)^{-1/2} \tag{C4a}$$

and that a similar distribution for the valence quarks is given by

$$dP_v(x) \sim x^{1-\alpha(0)}\left(x^2 + m^2/b^2\right)^{-1/2} dx \tag{C4b}$$

where the Regge form appropriate for inelastic scattering is used. With these expressions, the distribution for an n-quark state with three valence quarks and $n-3$ quarks and antiquarks in the core pairs becomes

$$dP(x_1,\ldots,x_n) = Z\left(g^k/k!\right)\delta\left(1 - \sum_{i=1}^{n} x_n\right) \times \prod_{i=1}^{3} x_i^{1-\alpha(0)} \prod_{j=1}^{n} dx_j\left(x_i^2 + m^2/b^2\right)^{-1/2} \tag{C4c}$$

where Z is a normalization constant and where $g^k/k!$ is a statistical factor. The functions $G_i(\omega)$ are found when (C4c) is integrated with the aid of an exponential representation of the δ function. In this way, one finds expressions for the spin averaged as well as the spin-dependent structure functions which depend upon the parameters $\alpha(0)$ and g. With a suitable choice for these parameters, one finds reasonable agreement with the experimental data.

A less well-known approach which also leads to scaling and to explicit representations of the structure functions for deep inelastic scattering is described in Schwinger (1975a, b; 1976a, b). In the remainder of this appendix, I give a brief description of this approach. For this method, the basic physical assumption is that information about the structure functions in the deep inelastic region can be obtained as the result of a smooth extrapolation of information from the resonance region for electron–nucleon scattering as well as from the region which pertains to photon absorption. The principal mathematical assumption is that the structure functions can be represented as a double spectral integral. This results in expressions of the form

$$H_i(4t,\nu) = \iint \frac{dM_+^2}{M_+^2}\frac{dM_-^2}{M_-^2} 2h_i\left(M_+^2, M_-^2\right)$$

$$\times\left[(q+b)^2 - M_+^2 + i\varepsilon\right]^{-1}\left[(q-b)^2 - M_-^2 + i\varepsilon\right]^{-1} \quad \text{(C5a)}$$

Applying the integral transformation

$$(1/x) = \int_0^\infty e^{-x\xi}\, d\xi$$

to the second term in the denominator and extracting the imaginary part, one finds after integration

$$\operatorname{Im} H_{1,2}(4t,\nu) = \frac{1}{\left(M_+^2\right)^2}\int_0^\infty d\xi \exp\left[\left(\frac{4t}{M_+^2}\right)\xi\right] h_{1,2}\left(\frac{\xi, m^2}{M_+^2}\right) \quad \text{(C5b)}$$

Information about the properties of the functions $h_i(\xi)$ may be obtained from the known behavior in the region near elastic scattering as well as from the region appropriate for photon absorption. In the first case, one can show that

$$F_3^2, \frac{F_e^2 - \left(t/m^2\right)F_m^2}{1 - t/m^2} = \int_0^\infty d\xi\, e^{(4t/M_+^2)\xi} h'_{1,2}(\xi) \quad \text{(C6a)}$$

Experimentally it is known that the functions $F_e(4t)$ and $F_m(4t)$ are reasonably represented by the function

$$F_i(4t) = F_i(0)\left(1 - 4t/m_0^2\right)^{-2} \quad \text{(C6b)}$$

with $m_0 = 0.9m$. For $-t \gg 0$, this yields the result

$$h'_{1,2}(\xi) \sim \xi^3$$

for $\xi < 1$. In passing, one should note that the same result is found for $h_3(\xi)$ and $h_4(\xi)$ if $H_3(4t,\nu)$ in (76a) is replaced with $(1 - t/m^2)H_3(4t,\nu)$. From an analysis of the high-energy behavior of the photon cross section, one concludes that a similar function $\bar{h}_2(\xi)$ is appropriate for this process and that it behaves as $\bar{h}_2(\xi) \sim 1$ for $\xi \gg 1$.

For deep inelastic scattering, both $2\nu/m$ and $-4t/m^2$ are large and $\omega > 1$. In this domain $M_+^2 \sim (\omega - 1)(-4t)$, and it follows from (C5b), (59a), and (61) that

$$\sigma = \sigma_T + \sigma_L = -(\pi\alpha/t) f_2(\omega) \quad \text{(C7a)}$$

with

$$f_2(\omega)=\frac{\omega}{\omega-1}\int_0^{\infty}\exp\left[-\frac{\xi}{(\omega-1)}\right]\bar{h}_2'(\xi)\,d\xi \qquad \text{(C7b)}$$

The extrapolation which connects the deep inelastic phenomena with the phenomena which occur in the resonance region is made if one identifies the function $\bar{h}_2(\xi)$ in (C7b) with the $h_2(\xi)$ that occurs in (C6). This suggests the replacement

$$-4t/m^2 \to (\omega-1)^{-1}$$

which is to be made in $F_e(4t)$ and $F_m(4t)$ which occur in (C6a) and (C6b). For electron–proton scattering, this leads to the result

$$f_P(\omega)\sim\frac{\omega(\omega-1)^3}{(\omega+0.2)^4}\frac{(\omega+0.95)}{(\omega-0.75)}\cdot g_i(\omega) \qquad \text{(C8)}$$

with

$$g_1(\omega)=1+1.4(\omega-0.75)^{-1/2}, \qquad \omega>3$$

$$g_2(\omega)=1+1.15(\omega-1)-0.34(\omega-1)^2, \qquad \omega<3 \qquad \text{(C9)}$$

The function $g_1(\omega)$ is suggested from the high-energy behavior of the photon absorption cross section, and it is assumed that this function is valid for values of ω down to 3. The function $g_2(\omega)$ results from a smooth quadratic extrapolation to the value $\omega=1$ such that $f_P(\omega)\sim(\omega-1)^3$ as $\omega\to1$. The functions $g_1(\omega)$ and $g_2(\omega)$ as well as their derivatives match at $\omega=3$. As in the case of the quark model, one finds with the present method that the structure function

$$W_2(\omega)=(1/\pi\nu)f_2(\omega) \qquad \text{(C10)}$$

is in good agreement with the experimental results when ω is replaced with an improved scaling variable. From the above discussion, it is clear that the quark model, although frequently used for the interpretation of the experimental results for deep inelastic scattering, is not the only physically appealing theoretical approach to the understanding of these phenomena.

NOTE ADDED IN PROOF

The methods developed in this paper have been applied to other polarization phenomena, and they have been extended to include the electroweak interactions of neutrinos and electrons. These applications may be found in:

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